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Universal holographic chiral dynamics in an external magnetic field

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Abstract: In this work we further extend the investigation of holographic gauge theories in external magnetic fields, continuing earlier work. We study the phenomenon of magnetic catalysis of mass generation in 1+3 and 1+2 dimensions, using D3/D7- and D3/D5-brane systems, respectively. We obtain the low energy effective actions of the corresponding pseudo Goldstone bosons and study their dispersion relations. The D3/D7 system exhibits the usual Gell-Mann–Oakes–Renner (GMOR) relation and a relativistic dispersion relation, while the D3/D5 system exhibits a quadratic non-relativistic dispersion relation and a modified linear GMOR relation. The low energy effective action of the D3/D5 system is related to that describing magnon excitations in a ferromagnet. We also study properties of general Dp/Dq systems in an external magnetic field and verify the universality of the magnetic catalysis of dynamical symmetry breaking.

Keywords: AdS-CFT Correspondence, D-branes

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Contents

1 Introduction

The applications of holographic gauge/gravity correspondences to the study of more and more diverse phenomena are ever widening in scope. Over the last half a decade the links between finite temperature generalizations of AdS/CFT and experimental heavy-ion collisions have become much more concrete and the theoretical methods available to us are yielding ever deeper results concerning the properties of the quark-gluon plasma (see ref. [1, 2] for a recent review). Moreover in the last year the links between holography and condensed matter systems have also flourished, with work on superconductivity, and superfluidity, quantum phase transitions and both the classical and quantum hall effects having recent successes (e.g. refs. [3, 4] and references therein).

In the present work we extend the investigation of holographic gauge theories in the presence of external magnetic fields from the work first studied in ref. [5]. In this paper we are interested in finding both universal properties of strongly coupled gauge theories in the presence of magnetic fields, as well as in the different phenomena exhibited in such theories in a variety of space-time dimensions.

The phenomenon of dynamical flavor symmetry breaking catalysed by an arbitrarily weak magnetic field is known from refs. [6, 7] and refs. [8–10]. This effect was shown to be model independent and therefore insensitive to the microscopic physics underlying the low energy effective theory. In particular the infra-red (IR) description of the Goldstone modes associated with the dynamically broken symmetry should be universal. We therefore expect to be able to study this phenomenon using the holographic formalism. The aim of the present study will be to investigate the dynamics of the Goldstone modes and construct the low energy chiral Lagrangian of theories both in 3+1 and 2+1 dimensions in the presence of external magnetic fields, showing that the appropriate holographic models give precisely the results expected from the traditional field theory approach.

The effective dynamics of fermion pairing, in $d+1$ dimensions, in the presence of an external magnetic field is constrained to $d-2$ spatial dimensions. For this reason there are marked differences in the phenomenology of such systems in two and three spatial dimensions. In 2+1 dimensions refs. [7]–[10] Poincare symmetry is broken by the magnetic field (there is no longer any trace of the original boost invariance), removing the strong constraints on the dynamics of Goldstone modes imposed by special relativity. The naive Goldstone boson counting therefore does not hold and the resulting dispersion relation for the Goldstone modes takes a quadratic form, unlike in the case of 3+1 dimensions where an $\mathrm{SO}(1,1)$ subgroup of $\mathrm{SO}(3,1)$ constrains the dynamics. Although the number of Goldstone particles is no longer constrained in the non-relativistic setting, the number of Goldstone fields is fixed by the dimension of G/H (G=symmetry of the action, H=symmetry of the ground state). We will show that this also holds in the AdS/CFT context.

The 2+1 dimensional model is of particular interest because, as shown in ref. [11], the low energy effective description is that of magnon excitations in a ferromagnet. Using a D3/D5 brane intersection we will be able to reproduce such a result at quadratic order in the chiral Lagrangian. Moreover such 2+1 dimensional theories may have relevance in the arenas of the quantum hall effect, and high T_c superconductivity.

In addition to the phenomena discussed specifically in two and three spatial dimensions we show that certain universal behaviors are exhibited holographically in the present context. Here we will study holographic systems T-dual to the D3/D7 flavor model and show that the existence of an arbitrarily small magnetic field induces a spiral behaviour in the equation of state for such systems. In the limit that the chiral symmetry of the underlying theory is preserved, this equation of state can be studied analytically and such a symmetric vacuum can be shown to be unstable. This is holographically equivalent to the findings of refs. [6]–[10] — flavor symmetry breaking is induced dynamically by the presence of a magnetic field.

The outline of the present paper is as follows.

In section 2 we will return to the D3/D7 brane intersection in the presence of an external magnetic field, discussed in ref. [5]. We shall show explicitly how the magnetic catalysis of flavor symmetry breaking is realised in the holographic system, including the calculation of the chiral Lagrangian to second order in the low energy degrees of freedom. We will show that the Gell-Mann–Oakes–Renner relation holds analytically and obtain the dispersion relation for the Goldstone modes.

In section 3 we turn to the case of the D3/D5 defect theory and show here how the SO(3) flavor symmetry is dynamically broken to the U(1) subgroup in the presence of a magnetic field. In this non-relativistic system we find the Goldstone modes and show that the number of massless modes is not the same as the number of broken generators, but satisfies a more general counting rule [24] applicable for non-relativistic systems. We also show that the single Goldstone mode satisfies a modified Gell-Mann–Oakes–Renner relation and a quadratic dispersion relation. Again we obtain the dispersion relation analytically in the small mass limit and find the low energy effective Lagrangian which describes magnon excitations in a ferromagnet.

In section 4 we prove that the magnetic catalysis of dynamical symmetry breaking is a universal effect in gauge theories dual to Dp/Dq intersections in the appropriate decoupling limit. This proof involves showing that all such systems exhibit a self-similar spiral behaviour in their equation of state which leads to an instability for the solution with zero dynamical mass. Just as in the work of ref. [7] this effect is independent of the magnitude of the external magnetic field.

2 Mass generation in the D3/D7 system

In this section we will review the results of refs. [5, 12, 13], where a holographic study of flavored $\mathcal{N} = 4$ supersymmetric Yang-Mills in an external magnetic field was studied using the D3/D7 system. We will focus on the effect of mass generation by magnetic catalysis in this theory and provide a detailed analysis of the pseudo-Goldstone mode associated to the spontaneous breaking of a global U(1) R-symmetry. In particular we will show that the Gell-Mann–Oakes–Renner relation for the mass of the corresponding η' meson is satisfied.

The D3/D7 system provides a dual holographic description of N_f fundamental $\mathcal{N} = 2$ hypermultiplets coupled to $\mathcal{N} = 4$ SU(N_c) supersymmetric Yang Mills theory in the quenched approximation $N_f \ll N_c$ [14]. At zero separation between the D3 and D7-branes the fundamental hypermultiplets are massless and the β-function of the theory is proportional to N_f/N_c. Thus in the quenched approximation the β-function vanishes and the corresponding gauge theory is conformal. The global SO(6) R-symmetry of the $\mathcal{N} = 4$ SYM theory is broken to an SU(2)$\times$U(1) R-symmetry, the U(1) corresponding to rotations in the 2-plane transverse to both the D3 and D7-branes. The left and right handed fermions of the hypermultiplet have opposite charges under this U$(1)_R$ and thus the formation of a fermionic condensate $\langle\bar{\psi}\psi\rangle$ would lead to the spontaneous breaking of this symmetry.

2.1 Spontaneous symmetry breaking

There are various ways in which one can study the breaking of the chiral symmetry holographically. This has been studied in the past by the deformation of AdS$_5 \times S^5$ by a field corresponding to a marginally irrelevant operator on the gauge theory side refs. [16–18]. In the present case however we will stimulate the formation of a condensate by turning on the magnetic components of the U(1) gauge field of the D7-branes $F_{\alpha\beta}$ (equivalent to exciting a pure gauge $B-$field in the supergravity background). This U(1) gauge field corresponds to the diagonal U(1) of the full U(N_f) gauge symmetry of the stack of D7-branes. Since the D7-branes wrap an infinite internal volume, the dynamics of the $\mathrm{U}(N_f)$ gauge field is frozen in the four dimensional theory and the $\mathrm{U}(N_f)$ gauge symmetry becomes a global flavor symmetry $\mathrm{U}(N_f) = \mathrm{U}(1)_B \times \mathrm{SU}(N_f)$. Therefore the U(1) gauge field that we consider corresponds to the gauged $\mathrm{U}(1)_B$ baryon symmetry and the magnetic field that we introduce couples to the baryon charge of the fundamental fields [26].

The problem thus boils down to studying embeddings of probe D7-branes in the $\mathrm{AdS}_5 \times S^5$ background parameterized as follows:

$$\begin{aligned} ds^2 &= \frac{\rho^2+L^2}{R^2}\left[-dx_0^2+dx_1^2+dx_2^2+dx_3^2\right]+\frac{R^2}{\rho^2+L^2}\left[d\rho^2+\rho^2 d\Omega_3^2+dL^2+L^2 d\phi^2\right] , \\ d\Omega_3^2 &= d\psi^2+\cos^2\psi d\beta^2+\sin^2\psi d\gamma^2, \\ g_s C_{(4)} &= \frac{u^4}{R^4} dx^0\wedge dx^1\wedge dx^2\wedge dx^3; \qquad e^\Phi = g_s; \qquad R^4 = 4\pi g_s N_c \alpha'^2 , \end{aligned} \tag{2.1}$$

where $\rho, \psi, \beta, \gamma$ and L, ϕ are polar coordinates in the transverse $\mathbb{R}^4$ and $\mathbb{R}^2$ planes respectively.

Here $x_{a=1\dots3}, \rho, \psi, \beta, \gamma$ parameterize the world volume of the D7-brane and the following ansatz is considered for its embedding:

$$\phi \equiv \text{const} , \qquad L \equiv L(\rho) ,$$

leading to the following induced metric on its worldvolume:

$$d\tilde{s} = \frac{\rho^2+L(\rho)^2}{R^2}\left[-dx_0^2+dx_1^2+dx_2^2+dx_3^2\right]+\frac{R^2}{\rho^2+L(\rho)^2}\left[(1+L'(\rho)^2)d\rho^2+\rho^2 d\Omega_3^2\right] . \tag{2.2}$$

The D7-brane probe is described by the DBI action:

$$S_{\mathrm{DBI}} = -N_f\mu_7 \int\limits_{\mathcal{M}_8} d^8\xi e^{-\Phi}\left[-\det(G_{ab}+B_{ab}+2\pi\alpha' F_{ab})\right]^{1/2} . \tag{2.3}$$

Here $\mu_7 = [(2\pi)^7\alpha'^4]^{-1}$ is the D7-brane tension, G_{ab} and B_{ab} are the induced metric and B-field on the D7-brane's world volume, while F_{ab} is its world-volume gauge field. A simple way to introduce a magnetic field is to consider a pure gauge B-field along the x_2, x_3 directions:

$$B^{(2)} = H dx_2 \wedge dx_3 . \tag{2.4}$$

Since B_{ab} and F_{ab} appear on equal footing in the DBI action, the introduction of such a B-field is equivalent to introducing an external magnetic field of magnitude $H/(2\pi\alpha')$ to the dual gauge theory.

Though the full solution of the embedding can only be calculated numerically, the large ρ behaviour (equivalently the ultraviolet (UV) regime in the gauge theory language) can be extracted analytically:

$$L(\rho) = m + \frac{c}{\rho^2} + \cdots . \tag{2.5}$$

As discussed in ref. [16], the parameters m (the asymptotic separation of the D7- and D3-branes) and c (the degree of bending of the D7-brane in the large ρ region) are related to

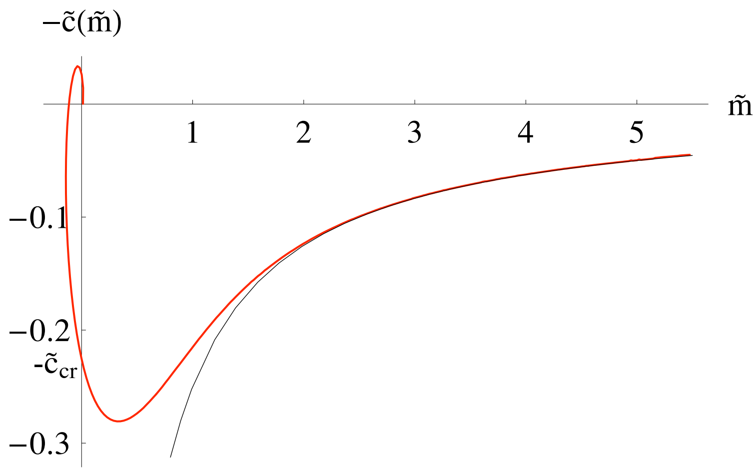

Figure 1. Parametric plot of $\tilde{c}$ against $\tilde{m}$ for fundamental matter in the presence of an external magnetic field. The lower (black) line represents the curve $1/\tilde{m}$, fitting the large $\tilde{m}$ behavior. It is also evident that for the outer branch of the spiral, for $\tilde{m} = 0$ the condensate, $\langle\bar{\psi}\psi\rangle$ is non-zero. The corresponding value of the condensate is $\tilde{c}_{\rm cr} = 0.226$.

the bare quark mass $m_q = m/2\pi\alpha'$ and the fermionic condensate $\langle\bar{\psi}\psi\rangle \propto -c$ respectively. It should be noted that the boundary behavior of $L(r)$ really plays the role of source and vacuum expectation value (vev) for the full $\mathcal{N} = 2$ hypermultiplet of operators. In the present case, where supersymmetry is broken by the gauge field configuration, we are only interested in the fermionic bilinears and this will refer only to quarks, and not their supersymmetric counterparts.

At this point it is convenient to introduce dimensionless parameters $\tilde{c} = c/R^3H^{3/2}$ and $\tilde{m} = m/R\sqrt{H}$. By performing a numerical shooting method from the infrared while varying the small ρ boundary value, $L(\rho \to 0) = L_{IR}$, we recover the parametric plot presented in figure 1, the main result explored in ref. [5].

The lower (black) curve corresponds to the analytic behavior of $\tilde{c}(\tilde{m}) = 1/\tilde{m}$ for large $\tilde{m}$. The most important observation is that at $\tilde{m} = 0$ there is a non-zero fermionic condensate:

$$\langle\bar{\psi}\psi\rangle = -\frac{N_f N_c}{(2\pi\alpha')^3\lambda} c = -\frac{N_f N_c \tilde{c}_{\rm cr}}{(2\pi^2)^{3/4}\lambda^{1/4}}\left(\frac{H}{2\pi\alpha'}\right)^{3/2} . \tag{2.6}$$

Where $\lambda = g^2_{\rm YM}N_c$ is the 't Hooft coupling and $\tilde{c}_{\rm cr} \approx 0.226$ is a numerical constant corresponding to the y-intercept of the outer spiral from figure 1. Equation (2.6) is telling us that the theory has developed a negative condensate that scales as $\left(\frac{H}{2\pi\alpha'}\right)^{3/2}$. This is not surprising, since the theory is conformal in the absence of the scale introduced by the external magnetic field. The energy scale controlled by the magnetic field, $\left(\frac{H}{2\pi\alpha'}\right)^{1/2}$, leads to an energy density proportional to $\left(\frac{H}{2\pi\alpha'}\right)^{2}$. In order to lower the energy, the theory responds to the magnetic field by developing a negative fermionic condensate.

Another interesting feature of the theory is the discrete-self-similar structure of the equation of state ($\tilde{c}$ *vs.* $\tilde{m}$) in the vicinity of the trivial $\tilde{m} = 0$ embedding, namely the origin of the plot from figure 1 presented in figure 2.

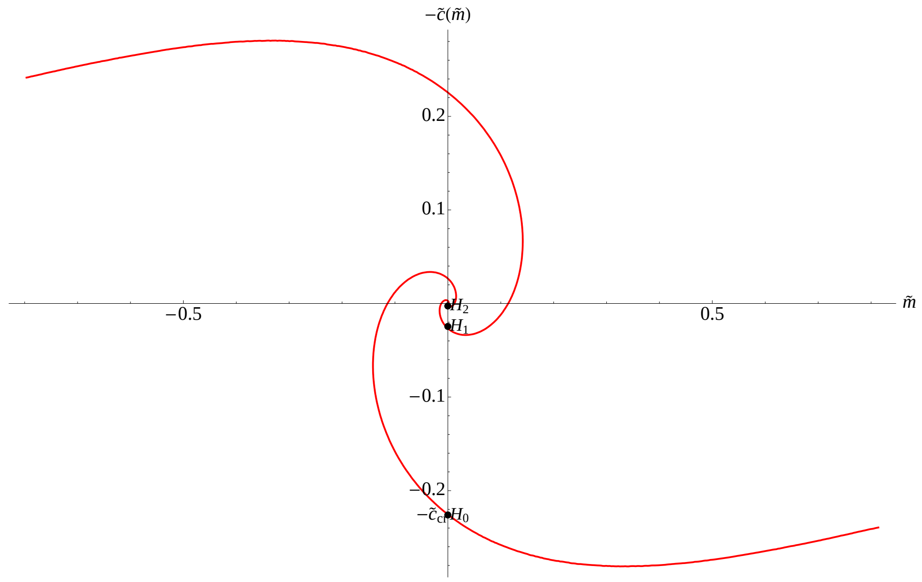

Figure 2. A magnification of figure 1 shows the spiral behavior near the origin of the $(-\tilde{c}, \tilde{m})$-plane. The second (left) spiral arm represents the $(\tilde{m}, -\tilde{c}) \rightarrow (-\tilde{m}, \tilde{c})$ symmetry of the theory.

This double logarithmic structure has been analyzed in ref. [12], where a study of the meson spectrum revealed that only the outer branch of the spiral is tachyon free and corresponds to a stable phase having spontaneously broken chiral symmetry. In section 3 of this paper we will show that an identical structure is also present for the D3/D5 system and in section 4 we will demonstrate that this structure is a universal feature of the magnetic catalysis of mass generation for gauge theories holographically dual to Dp/Dq intersections.

A further result of refs. [5, 12, 13] was the detailed analysis of the light meson spectrum of the theory. In ref. [5] it was shown that the introduction of an external magnetic field breaks the degeneracy of the spectrum studied in ref. [15]. This manifests itself as Zeeman splitting of the energy levels. In the limit of zero quark mass, the study also revealed the existence of a massless "η' meson" corresponding to the spontaneously broken $\mathrm{U}(1)_R$ symmetry. In the next subsection we will revisit the study of the meson spectrum of the theory and provide an analytic proof of the Gell-Mann–Oakes–Renner relation [23]:

$$M_\pi^2 = -\frac{2\langle\bar{\psi}\psi\rangle}{f_\pi^2} m_q \,, \tag{2.7}$$

in the spirit of the analysis performed in ref. [16].

2.2 The Gell-Mann–Oakes–Renner relation — an analytic derivation

In order to study the light meson spectrum of the theory one needs to consider the quadratic fluctuations of the D7-brane embedding and study the corresponding normal modes [15]. Technically one should consider the full supergravity action for the D7-branes:

$$S_{\text{tot}} = S_{\text{DBI}} + S_{\text{WZ}} \,, \tag{2.8}$$

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where $S_{\rm DBI}$ is given by equation (2.3) and the relevant part of the Wess-Zumino term is given by [5]:

$$S_{\rm WZ} = \frac{(2\pi\alpha')^2}{2}\mu_7 \int F_{(2)} \wedge F_{(2)} \wedge C_{(4)} + (2\pi\alpha')\mu_7 \int F_{(2)} \wedge B_{(2)} \wedge \tilde{P}\left[C_{(4)}\right] , \tag{2.9}$$

The next step is to consider fluctuations of the D7-brane in the transverse $\mathbb{R}^2$:

$$L = L_0(\rho) + 2\pi\alpha'\delta L \; ; \qquad \phi = 2\pi\alpha'\Phi\,, \tag{2.10}$$

and expand equation (2.8) to second order in α'. Note that with such an expansion we should also consider fluctuations of the U(1) gauge field on the D7-brane. As demonstrated in refs. [5, 19] the effect of the magnetic field will be to mix the equations of motion for the scalar and vector fluctuations. In particular Φ couples to the A_0 and A_1 components of the gauge field, while δL couples to the A_2 and A_3 components. The rest of the components of the vector field decouple and can be consistently set to zero. This splitting of the meson spectrum is a manifestation of the broken Lorentz symmetry. Indeed the external magnetic field breaks the SO(1,3) Lorentz symmetry down to SO(1,1) $\times$ SO(2) corresponding to boosts in the x_0, x_1 plane and rotations in the x_2, x_3 plane. Since the massless "pion" that we are interested in corresponds to fluctuations along ϕ, we will excite only the Φ, A_0, A_1 fields. The relevant terms of the expansion are [5]:

$$\begin{aligned}
\mathcal{L}_{\phi\phi} &= -(2\pi\alpha')^2\frac{\mu_7}{g_s}\frac{1}{2}\sqrt{|g_{S^3}|}\frac{gR^2L_0^2}{\rho^2+L_0^2}S^{ab}\partial_a\Phi\partial_b\Phi\,,\\
\mathcal{L}_{\Phi A} &= -(2\pi\alpha')^2\frac{\mu_7}{g_s}\sqrt{|g_{S^3}|}H\partial_\rho K\Phi F_{01}\,,\\
\mathcal{L}_{AA} &= -(2\pi\alpha')^2\frac{\mu_7}{g_s}\sqrt{|g_{S^3}|}\frac{1}{4}gS^{aa'}S^{bb'}F_{ab}F_{a'b'}\,,
\end{aligned} \tag{2.11}$$

where:

$$\left|\left|S^{ab}\right|\right| = \mathrm{diag}\left\{-G_{11}^{-1}, G_{11}^{-1}, \frac{G_{11}}{G_{11}^2+H^2}, \frac{G_{11}}{G_{11}^2+H^2}, G_{\rho\rho}^{-1}, G_{\psi\psi}^{-1}, G_{\alpha\alpha}^{-1}, G_{\beta\beta}^{-1}\right\} , \tag{2.12}$$

$$g(\rho) = \rho^3\sqrt{1+L_0'^2}\sqrt{1+\frac{R^4H^2}{(\rho^2+L_0^2)^2}}; \quad K(\rho) = \frac{R^4\rho^4}{(\rho^2+L_0^2)^2}; \quad \sqrt{|g_{S^3}|} = \sin\psi\cos\psi\,.$$

Here $L_0(\rho)$ corresponds to the classical embedding of the D7-brane and G_{ab} are the components of the background metric equation (2.1).

The equations of motions for Φ and F_{01} are calculated from the quadratic action, resulting in:

$$\begin{aligned}
\frac{1}{g(\rho)}\partial_\rho\left(\frac{g(\rho)L_0^2\partial_\rho\Phi}{1+L_0'^2}\right) + \frac{L_0^2\Delta_{\Omega_3}\Phi}{\rho^2} + \frac{R^4L_0^2}{(\rho^2+L_0^2)^2}\tilde{\Box}\Phi - \frac{H\partial_\rho K}{g(\rho)}F_{01} &= 0\,,\\
\frac{1}{g(\rho)}\partial_\rho\left(\frac{g(\rho)\partial_\rho F_{01}}{1+L_0'^2}\right) + \frac{\Delta_{\Omega_3}F_{01}}{\rho^2} + \frac{R^4}{(\rho^2+L_0^2)^2}\tilde{\Box}F_{01} - \frac{H\partial_\rho K}{g(\rho)}(-\partial_0^2+\partial_1^2)\Phi &= 0\,,
\end{aligned} \tag{2.13}$$

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where $F_{01} = \partial_0 A_1 - \partial_1 A_0$ and the gauge constraint $-\partial_0 A_0 + \partial_1 A_1 = 0$ is imposed (note that this is the usual Lorentz gauge, corresponding to the unbroken $\mathrm{SO}(1,1)$) and we have defined:

$$\widetilde{\Box} = -\partial_0^2 + \partial_1^2 + \frac{\partial_2^2 + \partial_3^2}{1 + \frac{R^4 H^2}{(\rho^2 + L_0^2)^2}} \ . \tag{2.14}$$

Once again the broken Lorentz symmetry is manifest in equation (2.14). The definition of the spectrum is now a subtle issue in the presence of the broken space-time symmetry. We will define the spectrum as the energy of a particle as measured in its rest frame. In fact because we retain the $\mathrm{SO}(1,1)$ symmetry we may consider fluctuations propagating in the x_1 direction. Since we are interested in describing the lowest lying modes ("pions" in particular) we will focus on modes that have no S^3 dependence. Therefore we consider the ansätze:

$$\Phi = e^{i(k_0 x^0 + k_1 x^1)} h(\rho); \qquad F_{01} = e^{i(k_0 x^0 + k_1 x^1)} f(\rho) \, , \tag{2.15}$$

and define:

$$M^2 = k_0^2 - k_1^2 \ . \tag{2.16}$$

The equations (2.13) simplify to:

$$\begin{aligned} \frac{1}{g}\partial_\rho \left(\frac{g L_0^2}{1 + L_0'^2} \partial_\rho h \right) + \frac{R^4 L_0^2}{(\rho^2 + L_0^2)^2} M^2 h - \frac{H \partial_\rho K}{g} f &= 0 \, , \\ \frac{1}{g}\partial_\rho \left(\frac{g}{1 + L_0'^2} \partial_\rho f \right) + \frac{R^4}{(\rho^2 + L_0^2)^2} M^2 f - \frac{M^2 H \partial_\rho K}{g} h &= 0 \ . \end{aligned} \tag{2.17}$$

Note that for large bare masses m (and correspondingly large values of L) the term proportional to the magnetic field is suppressed and the meson spectrum should approximate to the result for the pure $\mathrm{AdS}_5 \times S^5$ space-time case studied in ref. [15], where the authors obtained the following relation:

$$M_n = \frac{2m}{R^2} \sqrt{(n+1)(n+3)} \, , \tag{2.18}$$

between the eigenvalue of the n^{th} excited state ω_n and the bare mass m. If one imposes the boundary conditions:

$$h(\epsilon) = 1; \qquad h'(\epsilon) = 0; \qquad f(\epsilon) = 1; \qquad f'(\epsilon) = 0 \, , \tag{2.19}$$

the coupled system of differential equations can be solved numerically. Then by requiring the functions $h(\rho)$ and $f(\rho)$ to be regular at infinity one can quantize the spectrum of the fluctuations. It is also convenient to define the following dimensionless parameter $\tilde{M} = MR/\sqrt{H}$. The resulting plot for the first three excited states is presented in figure 3. There is Zeeman splitting of the states due to the magnetic field. (In the absence of the field there are three straight lines emanating from the origin; these are split to form six curves.) Also, at zero bare quark mass there is indeed a massless Goldstone mode, appearing at the end of the lowest curve. Furthermore the plot in figure 4 shows that for small bare quark mass one can observe a characteristic $\tilde{M} \propto \sqrt{\tilde{m}}$ dependence. In the

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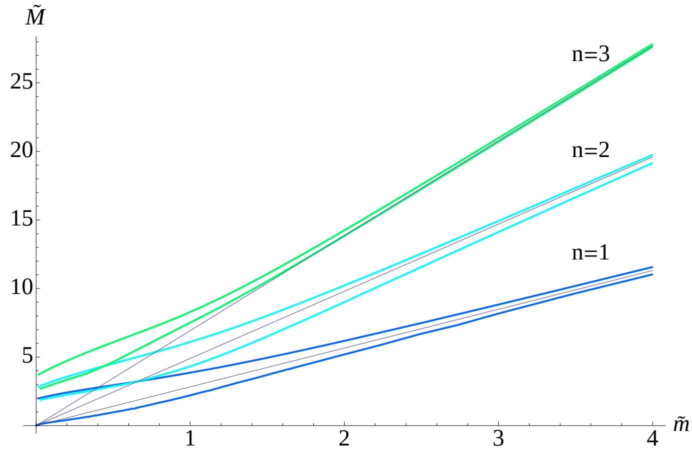

Figure 3. There is Zeeman splitting of the states due to the magnetic field. In the absence of the field there are three straight lines emanating from the origin; these are split to form six curves. At zero bare quark mass (the end of the lowest curve) there is indeed a massless Goldstone mode. The straight lines correspond to the asymptotic AdS results.

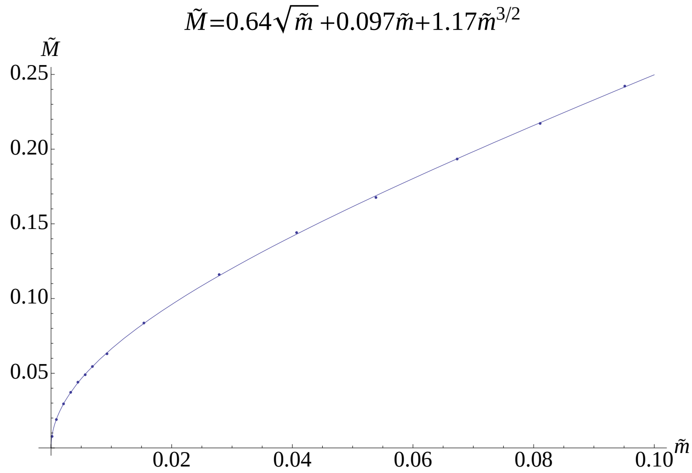

Figure 4. There is a characteristic $\tilde{M} \propto \sqrt{\tilde{m}}$ behavior at small bare quark mass.

next section we shall provide an analytic proof of that relation and obtain an integral expression for the numerical coefficient 0.64 presented above the plot in figure 4.

In the following section we shall demonstrate that for small bare quark mass, $m_q = m/2\pi\alpha'$, the spectrum exhibits the characteristic $M^2 \propto m$ dependence. Once we have illustrated that the functional dependence is correct we will show that the constant of proportionality is also that expected from the GMOR relation. Furthermore we shall generalize the ansätze (2.15) to consider fluctuations depending on both the momentum along the magnetic field $\vec{k}_{||} = (k_1, 0, 0)$ and the transverse momentum $\vec{k}_\perp = (0, k_2, k_3)$:

$$\Phi = e^{i(\omega t + \vec{k}.\vec{x})} h(\rho) \; ; \qquad F_{01} = e^{i(\omega t + \vec{k}.\vec{x})} f(\rho) \; . \tag{2.20}$$

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We shall also show that for small $\omega = k_0$ and $|\vec{k}|$ the following dispersion relation holds:

$$\omega(\vec{k})^2 = M^2 + \vec{k}_{||}^2 + \gamma \vec{k}_{\perp}^2 \ ; \quad \omega = k_0 \ ; \quad \vec{k}_{||} = (k_1, 0, 0) \ ; \quad \vec{k}_{\perp} = (0, k_2, k_3)\,, \tag{2.21}$$

where γ is a constant that we shall determine.

2.2.1 The $M^2 \propto m$ dependence

Using an approach similar to the one employed in ref. [16] we define:

$$\begin{aligned} &\Psi^2 = \frac{gL_0^2}{1+L_0'^2} \ ; \qquad \nu = R^4 \frac{1+L_0'^2}{(\rho^2+L_0^2)^2} \ ; \qquad \tilde{\nu} = R^4 \frac{1+L_0'^2}{(\rho^2+L_0^2)^2} \frac{1}{1+\frac{R^4H^2}{(\rho^2+L_0^2)^2}}\,, \\ &\Psi_1 = \Psi/L_0 \ ; \qquad \psi = h\Psi \ ; \qquad \psi_1 = f\Psi_1 \ . \end{aligned} \tag{2.22}$$

The equations of motions (2.13) can then be written in the compact form:

$$\begin{aligned} \ddot{\psi} - \frac{\ddot{\Psi}}{\Psi}\psi &= -\left(\omega^2 - \vec{k}_{||}^2\right)\nu\psi + \vec{k}_{\perp}^2\tilde{\nu}\psi + \frac{H\partial_\rho K}{\Psi\Psi_1}\psi_1\,, \\ \ddot{\psi}_1 - \frac{\ddot{\Psi}_1}{\Psi_1}\psi_1 &= -\left(\omega^2 - \vec{k}_{||}^2\right)\nu\psi_1 + \vec{k}_{\perp}^2\tilde{\nu}\psi_1 + \frac{H\partial_\rho K}{\Psi\Psi_1}\left(\omega^2 - \vec{k}_{||}^2\right)\psi \ . \end{aligned} \tag{2.23}$$

Let us remind the reader that for large ρ, $L_0(\rho)$ has the behavior:

$$L_0 \propto m + \frac{c}{\rho^2} + \cdots\,, \tag{2.24}$$

Let us denote by $\bar{L}_0$ the classical embedding corresponding to $(m = 0, c = c_{\rm cr})$. It is relatively easy to verify that at $m = 0, \vec{k}_{\perp} = \vec{0}$ and correspondingly $M^2 = \omega^2 - \vec{k}_{||}^2 = 0$ the choice:

$$\psi = \bar{\Psi} \equiv \Psi|_{\bar{L}_0} \ ; \qquad \psi_1 = 0\,, \tag{2.25}$$

is a solution to the system (2.23). Next we consider embeddings corresponding to a small bare quark mass δm. This will correspond to small nonzero values of M^2 and $\vec{k}_{\perp}^2$. It is then natural to consider the following variations:

$$\begin{aligned} \psi &= \bar{\Psi} + \delta\psi\,, \\ \psi_1 &= 0 + \delta\psi_1\,, \end{aligned} \tag{2.26}$$

where $\delta\psi$ and $\delta\psi_1$ are of order M^2. Note that M corresponds to the mass of the ground state at $m_q = \delta m/2\pi\alpha'$ and we are assuming that the variations of the wave functions $\delta\psi$ and $\delta\psi_1$ are infinitesimal for infinitesimal m_q. After expanding in equation (2.23) we get the following equations of motion:

$$\begin{aligned} \delta\ddot{\psi} - \frac{\ddot{\bar{\Psi}}}{\bar{\Psi}}\delta\psi - \delta\left(\frac{\ddot{\Psi}}{\Psi}\right)\bar{\Psi} &= -\left(\omega^2 - \vec{k}_{||}^2\right)\bar{\nu}\bar{\Psi} + \vec{k}_{\perp}^2\bar{\tilde{\nu}}\bar{\Psi} + \frac{H\partial_\rho K}{\bar{\Psi}_1\bar{\Psi}}\delta\psi_1\,, \\ \bar{\Psi}_1\delta\ddot{\psi}_1 - \ddot{\bar{\Psi}}_1\delta\psi_1 &= H\partial_\rho K\left(\omega^2 - k_{||}^2\right)\,, \end{aligned} \tag{2.27}$$

where $\bar{\nu} = \nu|_{\bar{L}_0}$. The second equation in (2.27) can be integrated to give:

$$\bar{\Psi}_1 \delta\dot{\psi}_1 - \dot{\bar{\Psi}}_1 \delta\psi_1 = HK\left(\omega^2 - k_{||}^2\right) + \text{constant} . \tag{2.28}$$

From the boundary conditions that $K|_{\rho=0} = 0$ and $\bar{\Psi}_1(0) = 0, \dot{\bar{\Psi}}_1(0) = 0$ we see that the constant of integration is zero and arrive at:

$$\partial_\rho \left(\frac{\delta\psi_1}{\bar{\Psi}_1}\right) = \frac{HK\left(\omega^2 - k_{||}^2\right)}{\bar{\Psi}_1^2} . \tag{2.29}$$

Next we multiply the first equation in (2.27) by $\bar{\Psi}$ and integrate along ρ to obtain:

$$\begin{aligned}\left(\omega^2 - \vec{k}_{||}^2\right)\int_0^\infty d\rho\bar{\nu}\bar{\Psi}^2 - \vec{k}_\perp^2\int_0^\infty d\rho\bar{\tilde{\nu}}\bar{\Psi}^2 &= -\int_0^\infty \left(\bar{\Psi}\delta\ddot{\psi} - \ddot{\bar{\Psi}}\delta\psi\right)d\rho + \int_0^\infty \bar{\Psi}^2\delta\left(\frac{\ddot{\Psi}}{\Psi}\right)d\rho + \\ &\quad + H\int_0^\infty \frac{\partial_\rho K\delta\psi_1}{\bar{\Psi}_1}d\rho \\ &= -\left(\bar{\Psi}\delta\dot{\psi} - \dot{\bar{\Psi}}\delta\psi\right)\Big|_0^\infty + \left(\bar{\Psi}\delta\dot{\Psi} - \dot{\bar{\Psi}}\delta\Psi\right)\Big|_0^\infty - H\int_0^\infty K\partial_\rho\left(\frac{\delta\psi_1}{\bar{\Psi}_1}\right)d\rho ,\end{aligned} \tag{2.30}$$

where the last term on the right-hand side of equation (2.30) has been integrated by parts using the fact that $\delta\psi_1$ should be regular at infinity. From the definition of $\bar{\Psi}$ it follows that $\bar{\Psi} \propto \rho^{3/2}L_0(0)$ as $\rho \to 0$ and $\bar{\Psi} \propto c/\rho^{1/2}$ as $\rho \to \infty$. This together with the requirement that ψ_1 is regular at $\rho = 0$ and vanishes at infinity, suggests that the first term on the right-hand side of equation (2.30) vanishes. For the next term, we use the fact that:

$$\delta\Psi = \rho^{3/2}\delta\left(\frac{1 + \frac{H^2R^4}{(\rho^2+L_0^2)^2}}{1 + L_0'^2}\right)^{1/4} L_0 + \rho^{3/2}\left(\frac{1 + \frac{H^2R^4}{(\rho^2+L_0^2)^2}}{1 + L_0'^2}\right)^{1/4}\delta L_0 , \tag{2.31}$$

and therefore obtain:

$$\begin{aligned} &\delta\Psi|_0 = 0; &&\delta\dot{\Psi}|_0 = 0 , \\ &\delta\Psi|_\infty \propto \rho^{3/2}\delta m ; &&\delta\dot{\Psi}|_\infty \propto \frac{3}{2}\sqrt{\rho}\delta m .\end{aligned} \tag{2.32}$$

The second term in equation (2.30) then becomes:

$$(\bar{\Psi}\delta\dot{\Psi} - \dot{\bar{\Psi}}\delta\Psi)\Big|_0^\infty = 2c\delta m . \tag{2.33}$$

Finally using the equality in equation (2.29) we arrive at the result:

$$\left(\omega^2 - \vec{k}_{||}^2\right)\int_0^\infty d\rho\left\{\bar{\nu}\bar{\Psi}^2 + \frac{H^2\bar{K}^2}{\bar{\Psi}_1^2}\right\} - \vec{k}_\perp^2\int_0^\infty d\rho\bar{\tilde{\nu}}\bar{\Psi}^2 = 2c\delta m . \tag{2.34}$$

Now we define:

$$\gamma = \left(\int_0^\infty d\rho \bar{\tilde{\nu}} \bar{\Psi}^2 \right) / \left(\int_0^\infty d\rho \left\{ \bar{\nu} \bar{\Psi}^2 + \frac{H^2 \bar{K}^2}{\bar{\Psi}_1^2} \right\} \right) , \tag{2.35}$$

and solve for M^2 from equation (2.21) to obtain:

$$M^2 \int_0^\infty d\rho \left\{ \bar{\nu} \bar{\Psi}^2 + \frac{H^2 \bar{K}^2}{\bar{\Psi}_1^2} \right\} = 2c\delta m \ . \tag{2.36}$$

Equation (2.36) suggests that the mass of the "pion" associated to the softly broken global U(1) symmetry satisfies the Gell-Mann–Oakes–Renner relation [23]:

$$M_\pi^2 = -\frac{2\langle \bar{\psi}\psi \rangle}{f_\pi^2} m_q \ . \tag{2.37}$$

In order to prove equation (2.37) we need to evaluate the effective coupling of the "pion" f_π^2. Noting that $\delta m \propto m_q$ and $c \propto -\langle \bar{\psi}\psi \rangle$, we conclude that:

$$f_\pi^2 \propto \int_0^\infty d\rho \left\{ \bar{\nu} \bar{\Psi}^2 + \frac{H^2 \bar{K}^2}{\bar{\Psi}_1^2} \right\} \ . \tag{2.38}$$

At this point is useful to verify the consistency of our analysis by comparing the coefficient in equation (2.36) to the numerically determined coefficient 0.64 from the plot in figure 4. Indeed from equation (2.36) we obtain:

$$\tilde{M}/\sqrt{\tilde{m}} = \left[\frac{1}{2\tilde{c}_{\rm cr}} \int_0^\infty d\tilde{\rho} \left\{ \bar{\hat{\nu}} \bar{\hat{\Psi}}^2 + \frac{\bar{\hat{K}}^2}{\bar{\hat{\Psi}}_1^2} \right\} \right]^{-1/2} \approx 0.655 \, , \tag{2.39}$$

where we have defined the dimensionless quantities:

$$\hat{\nu} = H^2 \nu; \qquad \hat{\Psi}^2 = \Psi^2 / R^5 H^{5/2}; \qquad \hat{\Psi}_1^2 = \Psi_1^2 / R^3 H^{3/2}; \qquad \hat{K} = K/R^4 \ . \tag{2.40}$$

There is excellent agreement with the fit from figure 4.

Next we will obtain an effective four dimensional action for the "pion" and from this derive an exact expression for f_π^2.

2.2.2 Effective chiral action and f_π^2

In this section we will reduce the eight dimensional world-volume action for the quadratic fluctuations of the D7-brane to an effective action for the massless "pion" associated to the spontaneously broken global U(1) symmetry. Note that our effective action should be describing a single "pion" mode, while the 8D action given by equation (2.11) describes the dynamics of two independent degrees of freedom, namely Φ and F_{01} coupled by the magnetic B-field *via* the second equation in equation (2.11). As rigid rotations along ϕ correspond to chiral rotations, (the asymptotic value of ϕ at infinity corresponds to the phase of the condensate in the dual gauge theory) the spectrum of Φ at zero quark mass

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contains the Goldstone mode that we are interested in. This is why we first integrate out the gauge field components A_0 and A_1 and then dimensionally reduce to four dimensions.

Furthermore as mentioned earlier, because of the magnetic field the SO(1,3) Lorentz symmetry is broken down to SO(1,1)×SO(2) symmetry. This is why in order to extract the value of f_π^2 we consider excitations of Φ depending only on the x_0, x_1 directions and read off the coefficient in front of the kinetic term. The resulting on-shell effective action for Φ is:

$$S^{\rm eff} = -\mathcal{N}\int d^4x \left[-(\partial_0\Phi)^2 + (\partial_1\Phi)^2\right] , \tag{2.41}$$

where $\mathcal{N}$ is given by:

$$\mathcal{N} = (2\pi\alpha')^2\frac{\mu_7}{g_s}N_f\pi^2\int\limits_0^\infty d\rho\left\{\bar{\nu}\bar{\Psi}^2 + \frac{H^2\bar{K}^2}{\bar{\Psi}_1^2}\right\} . \tag{2.42}$$

We refer the reader to appendix A for a detailed derivation of the 4D effective action $S^{\rm eff}$.

We have defined Φ *via* $\phi = (2\pi\alpha')\Phi$, where ϕ corresponds to rotations in the transverse $\mathbb{R}^2$ plane and is the angle of chiral rotation in the dual gauge theory. The chiral Lagrangian is then given by:

$$S^{\rm eff} = -(2\pi\alpha')^2\frac{f_\pi^2}{4}\int d^4x\partial_\mu\Phi\partial^\mu\Phi\ ; \qquad \mu = 0 \text{ or } 1\,, \tag{2.43}$$

and therefore:

$$f_\pi^2 = N_f 4\pi^2\frac{\mu_7}{g_s}\int\limits_0^\infty d\rho\left(\bar{\nu}\bar{\Psi}^2 + \frac{H^2K^2}{\bar{\Psi}_1^2}\right) . \tag{2.44}$$

The D7-brane charge in equation (2.44) is given by $\mu_7 = [(2\pi)^7\alpha'^4]^{-1}$ and the overall prefactor in equation (2.44) can be written as $N_fN_c/2(2\pi\alpha')^4\lambda$. Now, recalling the expressions for the fermionic condensate, equation (2.6), and the bare quark mass, $m_q = m/2\pi\alpha'$, one can easily verify that equation (2.36) is indeed the Gell-Mann–Oakes–Renner relation:

$$M_\pi^2 = -\frac{2\langle\bar{\psi}\psi\rangle}{f_\pi^2}m_q\ . \tag{2.45}$$

It turns out that for small momenta $\vec{k}_{||}, \vec{k}_\perp$ and small mass M_π^2 one can obtain the following more general effective 4D action (see appendix A for a detailed derivation):

$$S_{\rm eff} = -\mathcal{N}\int d^4x\left\{\left[-(\partial_0\tilde{\Phi})^2 + (\partial_1\tilde{\Phi})^2\right] + \gamma\left[(\partial_2\tilde{\Phi})^2 + (\partial_3\tilde{\Phi})^2\right] - \frac{2\langle\bar{\psi}\psi\rangle}{f_\pi^2}m_q\tilde{\Phi}^2\right\} + \cdots\,, \tag{2.46}$$

where γ is defined in equation (A.14). As one can see, the action (2.46) is the most general quadratic action consistent with the SO(1,1) × SO(2) symmetry and suggests that pseudo Goldstone bosons satisfy the dispersion relation (2.21).

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3 Mass generation in the D3/D5 system

In this section we provide a holographic description of the magnetic catalysis of chiral symmetry breaking in $1+3$ dimensional $\mathrm{SU}(N_c)$ $\mathcal{N}=4$ supersymmetric Yang-Mills theory coupled to N_f $\mathcal{N}=2$ fundamental hypermultiplets confined to a $1+2$ dimensional defect. Recently this theory received a great deal of attention and emphasis has been made of the potential application of this brane configuration in describing qualitative properties of $1+2$ dimensional condensed matter systems (see for example refs. [27–29]). In this section we will study the effect of an external magnetic field on the theory and demonstrate that the system develops a dynamically generated mass and negative fermionic condensate leading to a spontaneous breaking of a global SO(3) symmetry down to a U(1) symmetry. On the gravity side this symmetry corresponds to the rotational symmetry in the transverse $\mathbb{R}^3$. Naively there should be two massless Goldstone bosons corresponding to the generators of the coset SO(3)/U(1). As we will show the 1+2 dimensional nature of the defect theory leads to a coupling of the transverse scalars corresponding to the coset generators and as a result there is only a single Goldstone mode. Furthermore the characteristic $M_\pi \propto \sqrt{m}$ Gell-Mann–Oakes–Renner relation is modified to a linear $M_\pi \propto m$ behavior. It turns out that these features can be understood from a low energy effective theory point of view. Indeed in $1+2$ dimensions the effect of the magnetic field is to break the $\mathrm{SO}(1,2)$ Lorentz symmetry down to $\mathrm{SO}(2)$ rotational symmetry and as a result the theory is non-relativistic. A single time derivative chemical potential term is allowed (there is no boost symmetry) and interestingly the supergravity action generates such a term through the Wess-Zumino contribution of the D5-brane. It is this term that is responsible for the modified counting rule of the number of Goldstone bosons [24] and leads to a quadratic dispersion relation as well as to the modified linear Gell-Mann–Oakes–Renner relation. Another interesting feature of the model is that to quadratic order the effective low energy action is the same as the effective action describing spin waves in a ferromagnet [11] in an external magnetic field. We comment briefly on the possible applications of this similarity.

3.1 Generalities

Let us consider the $\mathrm{AdS}_5 \times S^5$ supergravity background (2.1) and introduce the following parameterization:

$$
\begin{aligned}
ds^2 &= \frac{u^2}{R^2}\left[-dx_0^2+dx_1^2+dx_2^2+dx_3^2\right]+\frac{R^2}{u^2}\left[dr^2+r^2d\Omega_2^2+dl^2+l^2d\tilde{\Omega}_2^2\right]\,, \qquad (3.1)\\
u^2 &= r^2+l^2\;; \qquad d\Omega_2^2 = d\alpha^2+\cos^2\alpha\, d\beta^2\;; \qquad d\tilde{\Omega}_2^2 = d\psi^2+\cos^2\psi\, d\phi^2\;.
\end{aligned}
$$

We have split the transverse $\mathbb{R}^6$ to $\mathbb{R}^3\times\mathbb{R}^3$ and introduced spherical coordinates r,Ω_2 and $l,\tilde{\Omega}_2$ in the first and second $\mathbb{R}^3$ planes respectively. Next we introduce a stack of probe N_f D5-branes extended along the x_0,x_1,x_2 directions, and filling the $\mathbb{R}^3$ part of the geometry parameterized by r,Ω_2. As mentioned above on the gauge theory side this corresponds to introducing N_f fundamental $\mathcal{N}=2$ hypermultiplets confined on a $1+2$ dimensional defect. The asymptotic separation of the D3 and D5 –branes in the transverse $\mathbb{R}^3$ space parameterized by l corresponds to the mass of the hypermultiplet. In the following we will consider the following anzatz for a single D5-brane:

$$l = l(r) \; ; \qquad \psi = 0 \; ; \qquad \phi = 0 \; . \tag{3.2}$$

The asymptotic separation $m = l(\infty)$ is related to the bare mass of the fundamental fields *via* $m_q = m/2\pi\alpha'$. If the D3 and D5 branes overlap, the fundamental fields in the gauge theory are massless and the theory has a global SO(3) $\times$ SO(3) symmetry. Clearly a non-trivial profile of the D5-brane $l(r)$ in the transverse $\mathbb{R}^3$ would break the global symmetry down to SO(3) $\times$ U(1), where U(1) is the little group in the transverse $\mathbb{R}^3$. If the asymptotic position of the D5-brane vanishes ($m = 0$) this would correspond to a spontaneous symmetry breaking, the non-zero separation $l(0)$ on the other hand would naturally be interpreted as the dynamically generated mass of the theory.

Note that the D3/D5 intersection is T-dual to the D3/D7 intersection from the previous section and thus the system is supersymmetric. The D3 and D5 –branes are BPS objects and there is no attractive potential for the D5-brane, hence the D5-brane has a trivial profile $l \equiv const$. However a non-zero magnetic field will break the supersymmetry and as we are going to demonstrate, the D5-brane will feel an effective repulsive potential that will lead to dynamical mass generation. In order to introduce a magnetic field perpendicular to the plane of the defect, we consider a pure gauge B-field in the x_1, x_2 plane given by:

$$B = H dx_1 \wedge dx_2 \; . \tag{3.3}$$

This is equivalent to turning on a non-zero value for the 0, 1 component of the gauge field on the D5-brane. The magnetic field introduced into the dual gauge theory in this way has a magnitude $H/2\pi\alpha'$. The D5-brane embedding is determined by the DBI action:

$$S_{\rm DBI} = -N_f \mu_5 \int\limits_{\mathcal{M}_6} d^6\xi e^{-\Phi} [-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})]^{1/2} \; . \tag{3.4}$$

Where G_{ab} and B_{ab} are the pull-back of the metric and the B-field respectively and F_{ab} is the gauge field on the D5-brane.

With the anzatz (3.2) the Lagrangian is given by:

$$\mathcal{L} \propto r^2 \sqrt{1 + l'^2} \sqrt{1 + \frac{R^4 H^2}{(r^2 + l^2)^2}} \; . \tag{3.5}$$

From this it is trivial to solve the equation of motion for $l(r)$ numerically, imposing $l(0) = l_{\rm in}$ and $l'(0)$ as initial conditions. Clearly, at large r the Lagrangian (3.5) asymptotes to that at zero magnetic field and hence we have the asymptotic solution [25]:

$$l(r) = m + \frac{c}{r} + \cdots \; , \tag{3.6}$$

where $c \propto \langle \bar{\psi}\psi \rangle$ the condensate of the fundamental fields.

3.2 Spontaneous symmetry breaking

Before solving the equation of motion it is convenient to introduce dimensionless variables:

$$\tilde{r} = r/R\sqrt{H} \ ; \quad \tilde{l} = l/R\sqrt{H} \ ; \quad \tilde{m} = m/R\sqrt{H} \ ; \quad \tilde{c} = c/R^2H \ . \tag{3.7}$$

The Lagrangian (3.5) can then be written as:

$$\mathcal{L} \propto \tilde{r}^2\sqrt{1+\tilde{l}'^2}\sqrt{1+\frac{1}{(\tilde{r}^2+\tilde{l}^2)^2}} \ . \tag{3.8}$$

The corresponding equation of motion is:

$$\partial_{\tilde{r}}\left(\frac{\tilde{r}^2 l'}{\sqrt{1+\tilde{l}'^2}}\frac{\sqrt{1+(\tilde{r}^2+\tilde{l}^2)^2}}{(\tilde{r}^2+\tilde{l}^2)}\right) = -2\frac{\tilde{r}^2\tilde{l}\sqrt{1+\tilde{l}'^2}}{(\tilde{r}^2+\tilde{l}^2)^2\sqrt{1+(\tilde{r}^2+\tilde{l}^2)^2}} \ . \tag{3.9}$$

Before solving equation (3.9) it will be useful to extract the asymptotic behavior of $\tilde{c}(\tilde{m})$ at large $\tilde{m}$. To this end we use that at large $\tilde{m}$ the separation $\tilde{l}(\tilde{r}) \approx \tilde{m} = \text{const}$. The equation of motion then simplifies to:

$$\partial_{\tilde{r}}(\tilde{r}^2\tilde{l}') = -\frac{2\tilde{r}^2\tilde{m}}{(\tilde{r}^2+\tilde{m}^2)^3} \ , \tag{3.10}$$

and hence:

$$\tilde{r}^2\tilde{l}' = -2\tilde{m}\int_0^{\tilde{r}} d\tilde{r}\frac{\tilde{r}^2}{(\tilde{r}^2+\tilde{m}^2)^2} \ . \tag{3.11}$$

Using the expansion (3.6) one can verify that:

$$\lim_{\tilde{r}\to+\infty}\tilde{r}^2\tilde{l}' = \tilde{c} = 2\tilde{m}\int_0^{\infty} d\tilde{r}\frac{\tilde{r}^2}{(\tilde{r}^2+\tilde{m}^2)^3} = \frac{\pi}{8\tilde{m}^2} \ . \tag{3.12}$$

Equation (3.12) can thus be used as a check of the accuracy of our numerical results. Indeed the numerically generated plot of $-\tilde{c}$ *vs.* $\tilde{m}$ is presented in figure 5. The most important observation is that at zero bare mass $\tilde{m}$ the theory has developed a negative condensate $\langle\bar{\psi}\psi\rangle \propto -\tilde{c}_{\text{cr}} \approx -0.59$. It can also be seen that for large $\tilde{m}$ the numerically generated plot is in good agreement with equation (3.12) represented by the lower (black) curve. Another interesting feature of the equation of state is the spiral structure near the origin of the parameter space analogous to the one presented in figure 2 for the case of the D3/D7 system. We will come back to this in section 4 in more general terms, and show that this feature is universal for the class of gauge theories dual to the Dp/Dq systems.

In order to show that the global SO(3) symmetry is indeed spontaneously broken we need to study the free energy of the theory. Indeed the existence of the spiral structure suggests that there is more than one phase at zero bare mass, corresponding to the different y-intercepts of the $-\tilde{c}$ *vs.* $\tilde{m}$ plot. We will demonstrate below that the lowest positive branch of the curve presented in figure 5 is the stable one.

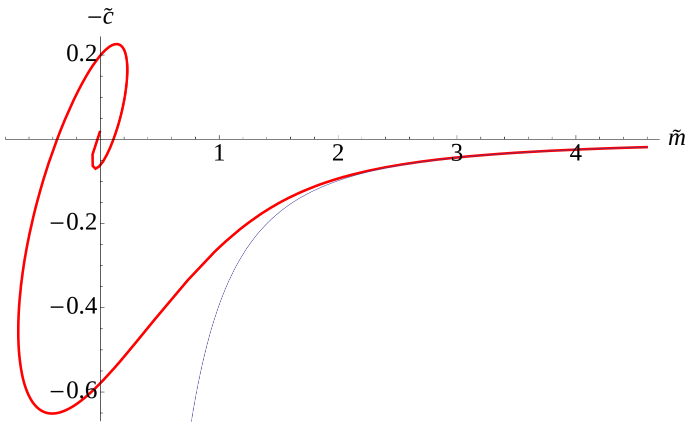

Figure 5. A plot of $-\tilde{c}$ *vs.* $\tilde{m}$. At zero bare mass $\tilde{m} = 0$ the theory has developed a negative condensate $\langle \bar{\psi}\psi \rangle \propto -\tilde{c}_{\rm cr} \approx -0.59$. For large $\tilde{m}$ there is excellent agreement with equation (3.12), as represented by the lower (blue) curve.

Following ref. [25] we will identify the regularized wick rotated on-shell action of the D5-brane with the free energy of the theory. Let us introduce a cut-off at infinity, $r_{\rm max}$, The wick rotated on-shell action is given by:

$$S = N_f \frac{\mu_5}{g_s} 4\pi V_3 R^3 H^{3/2} \int_0^{\tilde{r}_{\rm max}} d\tilde{r} \tilde{r}^2 \sqrt{1 + \tilde{l}'^2} \sqrt{1 + \frac{1}{(\tilde{r}^2 + \tilde{l}^2)^2}}\,, \tag{3.13}$$

where $V_3 = \int d^3x$ and $\tilde{l}(\tilde{r})$ is the solution of equation (3.9). It is easy to verify, using the expansion from equation (3.6), that the integral in equation (3.13) has the following behavior at large $\tilde{r}_{\rm max}$:

$$\int_0^{\tilde{r}_{\rm max}} d\tilde{r} \tilde{r}^2 \sqrt{1 + \tilde{l}'^2} \sqrt{1 + \frac{1}{(\tilde{r}^2 + \tilde{l}^2)^2}} = \frac{1}{3} r_{\rm max}^3 + O\left(\frac{1}{r_{\rm max}}\right)\,. \tag{3.14}$$

It is important that in these coordinates the divergent term is independent of the field $\tilde{l}$, it is therefore possible to regularize the on-shell action by subtracting the free energy of the $\tilde{l} \equiv 0$ embedding. The resulting regularized expression for the free energy is:

$$F = S_{\rm reg} = N_f \frac{\mu_5}{g_s} 4\pi V_3 R^3 H^{3/2} \tilde{I}_{\rm D5}\,, \tag{3.15}$$

where

$$\tilde{I}_{\rm D5} = \int_0^{\infty} d\tilde{r} \left[\tilde{r}^2 \sqrt{1 + \tilde{l}'^2} \sqrt{1 + \frac{1}{(\tilde{r}^2 + \tilde{l}^2)^2}} - \sqrt{1 + \tilde{r}^4} \right]\,. \tag{3.16}$$

A plot of $\tilde{I}_{\rm D5}$ *vs.* $|\tilde{m}|$ is presented in figure 6. The states from the lowest positive branch in figure 6 have the lowest free energy and correspond to the stable phase of the theory. Therefore there is a spontaneous breaking of the global SO(3) symmetry and the theory at $\tilde{m} = 0$ develops a negative condensate proportional to $-\tilde{c}_{\rm cr} \approx -0.59$. Note that

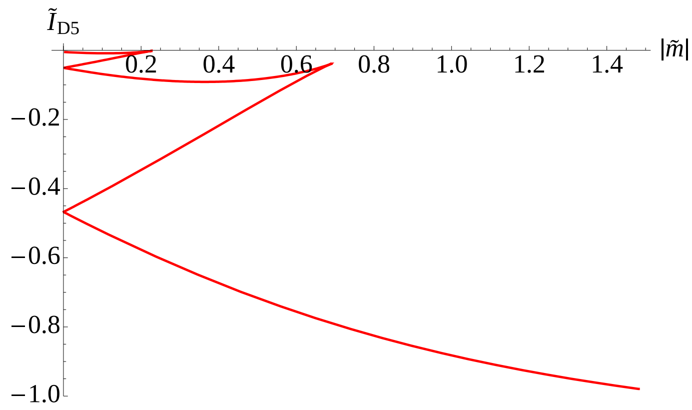

Figure 6. The states corresponding to the lowest positive branch of the plot in figure 5 have the lowest free energy and thus correspond to the stable phase of the theory.

only the absolute value of $\tilde{m}$ corresponds to the bare mass of the fundamental fields. The states with negative $\tilde{m}$ correspond to D5-brane embeddings that intercept the $\tilde{l} = 0$ line in the $\tilde{l}$ *vs.* $\tilde{r}$ plane and as seen from figure 6 are unstable. It is to be expected that the meson spectrum of the theory in such a phase would contain tachyons based on an analogy with the meson spectrum of the D3/D7 system studied in ref. [12]. Before we proceed with the analysis of the meson spectrum of the theory let us write an expression for the condensate of the theory $\langle\bar{\psi}\psi\rangle \propto -c_{\rm cr} = R^2 H\tilde{c}_{\rm cr}$ at zero bare quark mass. The coefficient of proportionality is given by [32]:

$$\langle\bar{\psi}\psi\rangle = -8\pi^2\alpha'\frac{\mu_5}{g_s}c_{\rm cr} = -16\pi^3\alpha'^2\frac{\mu_5}{g_s}\tilde{c}_{\rm cr}R^2(H/2\pi\alpha')\,. \tag{3.17}$$

Note that the condensate is proportional to the magnitude of the magnetic field $H/2\pi\alpha'$. In order to check the consistency of our numerical analysis and to calculate more accurately the constant $\tilde{c}_{\rm cr}$ we have calculated the value of $c_{\rm cr}$ for a range of H having set $R = 1$. The resulting plot is presented in figure 7. The solid (black) line corresponds to the linear fit $c_{\rm cr} \approx 0.586H$ therefore we have $\tilde{c}_{\rm cr} \approx 0.586$.

3.3 Meson spectrum and pseudo-Golstone bosons

In this section we will analyze the normal modes of the D5-brane. These describe fluctuations of the spinor bilinear in the dual gauge theory and hence their spectrum is the spectrum of the light meson-like excitations of the gauge theory. We focus our analysis on the normal modes corresponding to the Goldstone bosons (which we label as pions here for simplicity) of the spontaneously broken SO(3) symmetry and study their spectrum as a function of the bare quark mass m_q. Our study shows that the external magnetic field splits the degeneracy of the meson spectrum and gives mass to one of the pions of the theory. It also modifies the standard $M_\pi^2 \propto m$ GMOR relation for the remaining Goldstone mode to a linear relation $M_\pi \propto m$. We will show that these results are in accord with the behavior expected from the effective chiral Lagrangian of the theory.

In order to study the light meson spectrum of the theory we look for the quadratic fluctuations of the D5-brane embedding along the transverse directions parametrized by

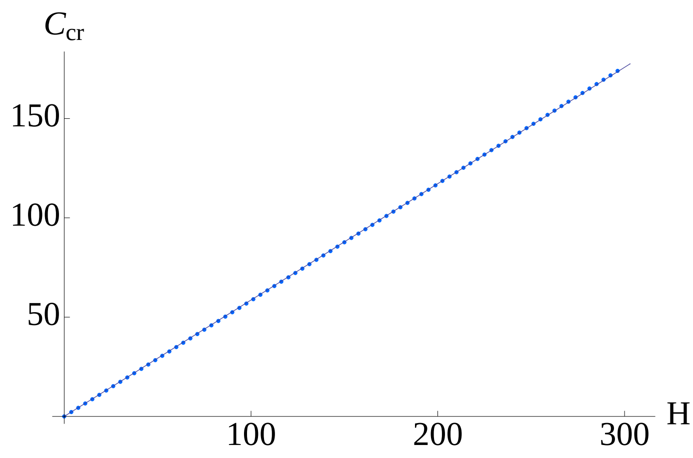

Figure 7. A plot of $c_{\rm cr}$ *vs.* H for $R = 1$. The solid (black) line corresponds to the linear fit $c_{\rm cr} \approx 0.586H$.

l, ψ, ϕ. To this end we expand:

$$l = \bar{l} + 2\pi\alpha'\delta l; \qquad \psi = 2\pi\alpha'\delta\psi; \qquad \phi = 2\pi\alpha'\delta\phi\,, \tag{3.18}$$

in the action (3.4) and leave only terms of order $(2\pi\alpha')^2$. Note that fluctuations of the U(1) gauge field $F_{\alpha\beta}$ of the D5-brane will also contribute to the expansion. There is also an additional contribution from the Wess-Zumino term of the D5-brane's action:

$$S_{\rm WZ} = N_f\mu_5 \int\limits_{\mathcal{M}_6} \sum_p \left[C_p \wedge e^{\mathcal{F}}\right]; \qquad \mathcal{F} = B + 2\pi\alpha' F\,. \tag{3.19}$$

For the anzatz that we are considering, the relevant term is:

$$S_{\rm WZ} = N_f\mu_5 \int\limits_{\mathcal{M}_6} B \wedge P\left[\tilde{C}_4\right]\,, \tag{3.20}$$

where $P[\tilde{C}_4]$ is the pull-back of the magnetic dual, $\tilde{C}_4$, to the background C_4 R-R form. For the particular parameterization of S_5 considered here, it is given by:

$$\tilde{C}_4 = \frac{1}{g_s}\frac{4r^2l^2}{(r^2+l^2)^3}R^4 \sin\psi(ldr - rdl)\wedge d\Omega_2 \wedge d\phi\,. \tag{3.21}$$

After some long but straightforward calculations we get the following action for the quadratic fluctuations along l:

$$\begin{aligned}
\mathcal{L}^{(2)}_{ll} &\propto \frac{1}{2}\sqrt{-E}G_{ll}\frac{S^{\alpha\beta}}{1+l'^2}\partial_\alpha\delta l\partial_\beta\delta l + \frac{1}{2}\left[\partial_l^2\sqrt{-E} - \frac{d}{dr}\left(\frac{l'}{1+l'^2}\partial_l\sqrt{-E}\right)\right]\delta l^2\,,\\
\mathcal{L}^{(2)}_{lF} &\propto \frac{\sqrt{-E}}{1+l'^2}(\partial_l J^{12} - \partial_r J^{12}l')F_{21}\delta l\,,\\
\mathcal{L}^{(2)}_{FF} &\propto \frac{1}{4}\sqrt{-E}S^{\alpha\beta}S^{\gamma\lambda}F_{\beta\gamma}F_{\alpha\lambda}\,,
\end{aligned} \tag{3.22}$$

and along ϕ and ψ:

$$\mathcal{L}^{(2)}_{\psi\psi,\phi\phi} \propto \frac{1}{2}\sqrt{-E}S^{\alpha\beta}(G_{\psi\psi}\partial_\alpha\delta\psi\partial_\beta\delta\psi + G_{\phi\phi}\partial_\alpha\delta\phi\partial_\beta\delta\phi) \; ,$$
$$\mathcal{L}^{(2)}_{\psi\phi} \propto (\cos\alpha)PH\delta\psi\partial_0\delta\phi \; . \tag{3.23}$$

Here $E_{\alpha\beta}$ is the pull-back of the generalized metric on the classical D5-brane embedding:

$$E_{\alpha\beta} = \partial_\alpha\bar{X}^\mu\partial_\beta\bar{X}^\nu(G_{\mu\nu} + B_{\mu\nu}) \, , \tag{3.24}$$

and we have defined $S^{\alpha\beta}$ and $J^{\alpha\beta}$ as the symmetric and anti-symmetric elements of the inverse generalized metric $E^{\alpha\beta}$:

$$E^{\alpha\beta} = S^{\alpha\beta} + J^{\alpha\beta} \; . \tag{3.25}$$

The determinant E and the function $K = P$ are given by:

$$\sqrt{-E} \;=\; (\cos\alpha)r^2\sqrt{1+l'^2}\sqrt{1+\frac{R^4H^2}{(r^2+l^2)^2}} \equiv g(r)\cos\alpha \, , \tag{3.26}$$

$$P \;=\; \frac{4R^4r^2l^2}{(r^2+l^2)^3}(rl'-l) \; . \tag{3.27}$$

As one can see, the fluctuations along ψ and ϕ decouple from the fluctuations along l and the fluctuations of the gauge field A_α. Since we are interested in the pseudo-Goldstone modes of the dual theory we will focus on the fluctuations along ψ and ϕ. The equations of motion derived from the quadratic action (3.23) are the following:

$$\partial_r\left(\frac{g(r)l^2}{1+l'^2}\partial_r\delta\psi\right) + \frac{g(r)R^4l^2}{(r^2+l^2)^2}\tilde{\Box}\delta\psi + \frac{g(r)R^4l^2r^2}{(r^2+l^2)^2}\Delta_{(2)}\delta\psi - PH\partial_0\delta\phi = 0 \, ,$$
$$\partial_r\left(\frac{g(r)l^2}{1+l'^2}\partial_r\delta\phi\right) + \frac{g(r)R^4l^2}{(r^2+l^2)^2}\tilde{\Box}\delta\phi + \frac{g(r)R^4l^2r^2}{(r^2+l^2)^2}\Delta_{(2)}\delta\phi + PH\partial_0\delta\psi = 0 \, , \tag{3.28}$$

where

$$\tilde{\Box} = -\partial_0^2 + \frac{\partial_1^2+\partial_2^2}{1+\frac{R^4H^2}{(r^2+l^2)^2}} \; . \tag{3.29}$$

Note that the background magnetic field breaks the SO(1, 2) Lorentz symmetry to SO(2), which manifests itself in the modified laplacian (3.29). Next we consider a plane-wave ansatz:

$$\delta\phi = e^{i(\omega t - \vec{k}\vec{x})}\eta_1(r); \qquad \delta\psi = e^{i(\omega t - \vec{k}\vec{x})}\eta_2(r) \, , \tag{3.30}$$

now using the anzatz (3.30) we get:

$$\partial_r\left(\frac{g(r)l^2}{1+l'^2}\eta_1'\right) + \frac{g(r)R^4l^2}{(r^2+l^2)^2}\left(\omega^2 - \frac{\vec{k}^2}{1+\frac{R^4H^2}{(r^2+l^2)^2}}\right)\eta_1 - i\omega PH\eta_2 = 0 \, ,$$
$$\partial_r\left(\frac{g(r)l^2}{1+l'^2}\eta_2'\right) + \frac{g(r)R^4l^2}{(r^2+l^2)^2}\left(\omega^2 - \frac{\vec{k}^2}{1+\frac{R^4H^2}{(r^2+l^2)^2}}\right)\eta_2 + i\omega PH\eta_1 = 0 \; . \tag{3.31}$$

The equations of motion in (3.31) can be decoupled by the definition $\eta_\pm = \eta_1 \pm i\eta_2$. The result is:

$$\begin{aligned}
&\partial_r\left(\frac{g(r)l^2}{1+l'^2}\eta'_+\right)+\frac{g(r)R^4l^2}{(r^2+l^2)^2}\left(\omega^2-\frac{\vec{k}^2}{1+\frac{R^4H^2}{(r^2+l^2)^2}}\right)\eta_+-\omega PH\eta_+=0\,,\\
&\partial_r\left(\frac{g(r)l^2}{1+l'^2}\eta'_-\right)+\frac{g(r)R^4l^2}{(r^2+l^2)^2}\left(\omega^2-\frac{\vec{k}^2}{1+\frac{R^4H^2}{(r^2+l^2)^2}}\right)\eta_-+\omega PH\eta_-=0\,. \qquad (3.32)
\end{aligned}$$

Because of the broken Lorentz symmetry, the $1+2$ dimensional mass $M^2 = \omega^2 - \vec{k}^2$ depends on the choice of frame. We can define the spectrum of excitations as the rest energy (consider the frame with $\vec{k} = 0$) and as we shall observe, the spectrum is discrete. Furthermore just as in the D3/D7 case there is a Zeeman splitting of the spectrum due to the external magnetic field. Interestingly, at low energy the splitting is breaking the degeneracy of the lowest energy state and as a result there is only one pseudo-Goldstone boson. Note that this is not in contradiction with the Goldstone theorem because there is no Lorentz symmetry. This opens the possibility of having two types of Goldstone modes: type I and type II satisfying odd and even dispersion relations correspondingly. In this case there is a modified counting rule (ref. [24], see also ref. [30]) which states that *the number of GBs of type I plus twice the number of GBs of type II is greater than or equal to the number of broken generators.* As we are going to show below the single Goldstone mode that we see satisfies a quadratic dispersion relation (hence is type II) and the modified counting rule is not violated. Note also that for large bare masses m (and correspondingly large values of l) the term proportional to the magnetic field is suppressed and the meson spectrum should approximate to the result for the pure $\mathrm{AdS}_5 \times S^5$ space-time case studied in refs. [20, 21], where the authors obtained the following relation:

$$\omega_n = \frac{2m}{R^2}\sqrt{(n+1/2)(n+3/2)}\,, \qquad (3.33)$$

between the eigenvalue of the n^{th} excited state ω_n and the bare mass m.

In order to obtain the meson spectrum, we numerically solve the equations of motion (3.32) in the rest frame ($\vec{k} = 0$). The quantization condition for the spectrum comes from imposing regularity at infinity. More precisely we require that $\eta_\pm \sim 1/r$ at infinity ($r \to \infty$). The results are summarized in figure 8. Just as in the D3/D7 case we have defined the dimensionless quantities $\tilde{m} = m/R\sqrt{H}$ and $\tilde{\omega} = \omega R/\sqrt{H}$. As one can see from figure 8, for large $\tilde{m}$ the spectrum asymptotes to that of pure $\mathrm{AdS}_5 \times S^5$, given by equation (3.33). The Zeeman splitting of the spectrum is also evident. It is interesting that as a result of the splitting of the ground state there is only a single pseudo-Goldstone mode. Furthermore, as can be seen from figure 9, for small bare masses instead of the usual Gell-Mann–Oakes–Renner relation we obtain a linear dependence $\tilde{\omega} \sim \tilde{m}$. As we will show in the next subsection the slope is given by the relation:

$$\tilde{\omega} = \frac{4\tilde{c}_{\text{cr}}}{\pi}\tilde{m} \approx 0.736\tilde{m}\,. \qquad (3.34)$$

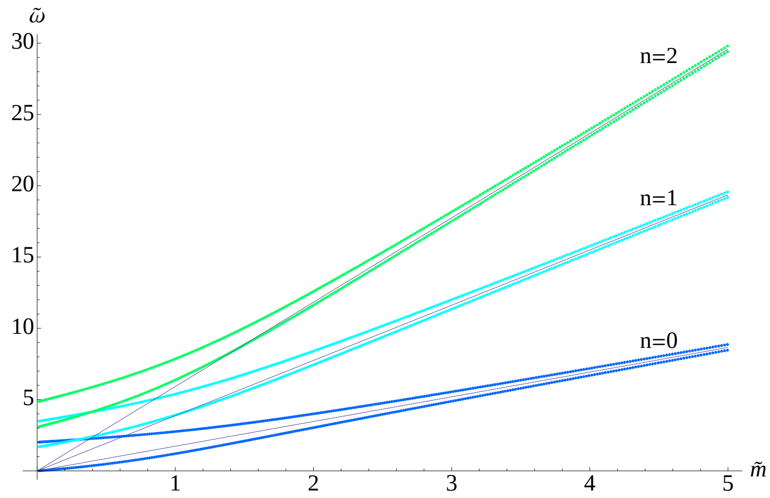

Figure 8. The meson spectrum of the first three excited states is plotted. There is Zeeman splitting of the spectrum and the existence of a mass gap at $\tilde{m} = 0$ as well as a single Goldstone boson mode. For large $\tilde{m}$ the spectrum asymptotes to that of zero magnetic field given by equation (3.33) (straight lines).

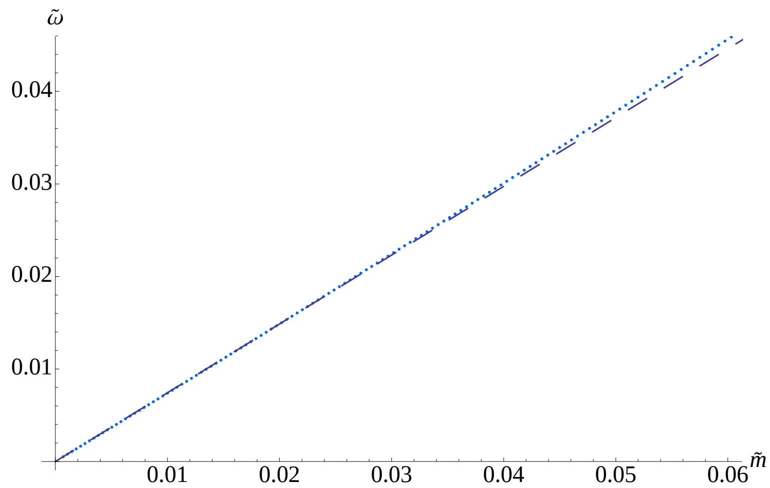

Figure 9. Plot of the spectrum of the ground state from figure 8 for small bare masses. The dashed line corresponds to the linear behavior from equation (3.34).

It is also interesting to study the dispersion relation of the Goldstone mode. Since we have broken Lorentz symmetry and observe only one pseudo-Goldstone mode (which is only half the number of broken generators) we anticipate a quadratic dispersion relation (see refs. [30] and [31] for discussion).

In order to obtain the dispersion relation of the Goldstone mode we numerically solve equations (3.32) at very small bare mass $\tilde{m} \approx 0.0007$ and for a range of small momenta $\tilde{\vec{k}} = \vec{k}R/\sqrt{H}$. The result is presented in figure 10. There is indeed a quadratic dispersion

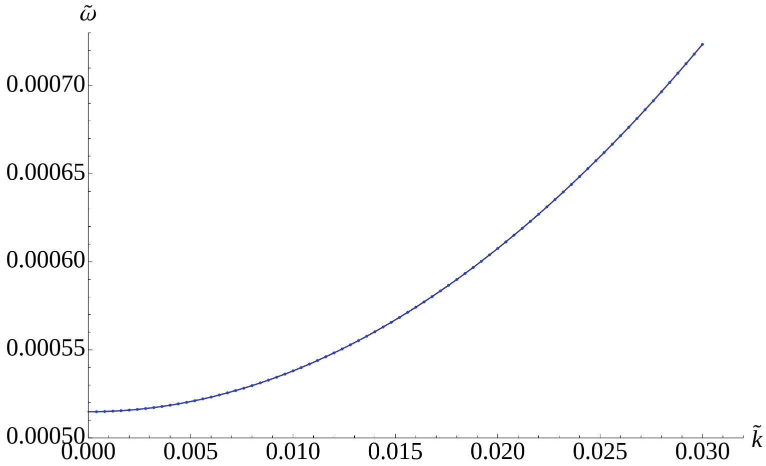

Figure 10. Plot of the dispersion relation of the pseudo-Goldstone mode for $\tilde{m} \approx 0.0007$. The parabolic fit corresponds to equation (3.37).

relation. As we are going to show, the dispersion relation is given by:

$$\tilde{\omega} = \gamma \tilde{\vec{k}}^2 + \frac{4}{\pi}\tilde{c}_{\rm cr}\tilde{m}\,, \tag{3.35}$$

where:

$$\gamma = \frac{4}{\pi}\int\limits_0^\infty d\tilde{r}\frac{\tilde{r}^2\tilde{l}^2\sqrt{1+\tilde{l}'^2}}{(\tilde{r}^2+\tilde{l}^2)\sqrt{1+(\tilde{r}^2+\tilde{l}^2)^2}}\,. \tag{3.36}$$

For $\tilde{m} \approx 0.0007$ the relation (3.35) is given by:

$$\tilde{\omega} \approx 0.232\tilde{\vec{k}}^2 + 0.000515\,, \tag{3.37}$$

and is represented by the fitted curve in figure 10.

In the next subsections we will obtain the effective $1+2$ dimensional chiral action describing the pseudo-Goldstone mode and argue that in the limit $\omega \to 0$ it is identical to the action describing magnon excitations in a ferromagnet [11]. Furthermore we will show that the observed dispersion relation (3.35) is in agreement with the dispersion relation of magnons in an external magnetic field. Note that in order to make the analogy with a ferromagnet, one needs to identify the bare mass with the external magnetic field acting on the ferromagnet. The reason is that these both correspond to the small parameter that explicitly breaks the global symmetry.

3.3.1 Low energy dispersion relation

In order to obtain the dispersion relation for the pseudo-Goldstone mode we will analyze the first equation in (3.32) in the spirit of the analysis performed in section 2.2.1 for the D3/D7 system. To begin with let us consider the limit of small ω thus leaving only the linear potential term in $\tilde{\omega}$. In view of the observed quadratic dispersion relation (3.35) we will also keep the $\vec{k}^2$ term in equation (3.32).

$$\partial_r \left(\frac{g(r)l^2}{1+l'^2} \eta'_+ \right) - \left(\omega P H + \frac{g(r)R^4 l^2}{(r^2+l^2)^2 + R^4 H^2} \vec{k}^2 \right) \eta_+ = 0 \,. \tag{3.38}$$

It is convenient to define the following variables:

$$\Theta^2 = \frac{g(r)l^2}{1+l'^2} \; ; \qquad \xi = \eta_+ \Theta \; . \tag{3.39}$$

Then equation (3.38) can be written as:

$$\ddot{\xi} - \frac{\ddot{\Theta}}{\Theta}\xi - \left(\omega P H + \frac{g(r)R^4 l^2}{(r^2+l^2)^2 + R^4 H^2} \vec{k}^2 \right) \frac{\xi}{\Theta^2} = 0 \; . \tag{3.40}$$

Where the overdots represent derivatives with respect to r. Now if we take the limit $m \to 0$ we have that $\omega \to 0$ and $k \to 0$ and obtain that:

$$\xi = \Theta|_{\omega=0} \equiv \bar{\Theta} \,, \tag{3.41}$$

is a solution to equation (3.40). Our next step is to consider small m and expand:

$$\xi = \bar{\Theta} + \delta\xi \; ; \qquad \Theta = \bar{\Theta} + \delta\Theta \,, \tag{3.42}$$

where the variations $\delta\xi$ and $\delta\Theta$ are vanishing in the $m \to 0$ limit. Then, to leading order in m (keeping in mind that $\omega \sim m$ and $\vec{k}^2 \sim m$) we obtain:

$$\delta\ddot{\xi} - \frac{\ddot{\bar{\Theta}}}{\bar{\Theta}}\delta\xi - \delta\left(\frac{\ddot{\Theta}}{\Theta} \right) \bar{\Theta} - \left(\omega P H + \frac{g(r)R^4 l^2}{(r^2+l^2)^2 + R^4 H^2} \vec{k}^2 \right) \frac{1}{\bar{\Theta}} = 0 \; . \tag{3.43}$$

Now we multiply equation (3.43) by $\bar{\Theta}$ and integrate along r. The result is:

$$(\bar{\Theta}\delta\dot{\xi} - \dot{\bar{\Theta}}\delta\xi)\Big|_0^\infty - (\bar{\Theta}\delta\dot{\Theta} - \dot{\bar{\Theta}}\delta\Theta)\Big|_0^\infty - \omega H \int_0^\infty dr P(r) - \frac{\pi}{4} R^5 \sqrt{H} \gamma \vec{k}^2 = 0 \; . \tag{3.44}$$

Using the definitions of $\Theta, P(r)$ and ξ and requiring regularity at infinity for η_+, one can show that the first term in equation (3.44) vanishes and that:

$$(\bar{\Theta}\delta\dot{\Theta} - \dot{\bar{\Theta}}\delta\Theta)\Big|_0^\infty = c\delta m \; ; \qquad \int_0^\infty dr P(r) = -R^4\pi/4 \,, \tag{3.45}$$

and hence using the previous definitions, $\tilde{m} = m/R\sqrt{H}$, $\tilde{c} = c/R^2 H$, $\tilde{\omega} = \omega R/\sqrt{H}$ and $\tilde{\vec{k}} = \vec{k}R/\sqrt{H}$, we obtain equation (3.35) which we duplicate below:

$$\tilde{\omega} = \gamma \tilde{\vec{k}}^2 + \frac{4}{\pi} \tilde{c}_{\rm cr} \tilde{m} \; . \tag{3.46}$$

In the next subsection we will derive the effective $1+2$ dimensional action for the pseudo-Goldstone mode and show that to leading order it is in to one correspondence with the effective action describing magnon excitations in a ferromagnet corresponding to the SO(3) $\to$ SO(2) spontaneous symmetry breaking by spontaneous magnetization [11]. We will relate the fermionic condensate $\tilde{c}$ to the spontaneous magnetization of the ferromagnet and the bare mass to the external magnetic field and show that the dispersion relation (3.46) is in exact agreement with that of magnons.

3.3.2 Effective chiral Lagrangian

In order to obtain the $1+2$ dimensional effective action describing the pseudo-Goldstone mode we consider the $1+5$ dimensional action (3.23) for a classical embedding in the vicinity of the critical embedding, namely that embedding corresponding to a very small bare mass $\tilde{m}$. Now let us consider the following ansätze for the fields $\delta\phi$ and $\delta\psi$:

$$\delta\phi = \frac{\xi_1(r)}{\Theta(r)}\chi_1(x)\ ; \qquad \delta\psi = \frac{\xi_2(r)}{\Theta(r)}\chi_2(x)\ . \tag{3.47}$$

Since we are close to the critical embedding we will consider the same expansion as in equation (3.42):

$$\xi_i = \bar\Theta + \delta\xi_i\,, \qquad i = 1\,\mathrm{or}\,2\ ; \qquad \Theta = \bar\Theta + \delta\Theta\ . \tag{3.48}$$

By definition it follows that as $\tilde{m} \to 0$, $\delta\xi_i$ and $\delta\Theta$ vanish. Then to leading order we have that:

$$\begin{aligned} \partial_r\delta\phi &= \frac{1}{\bar\Theta^2}[(\bar\Theta\delta\dot\xi_1 - \dot{\bar\Theta}\delta\xi_1) + (\dot{\bar\Theta}\delta\Theta - \bar\Theta\delta\dot\Theta)]\chi_1(x)\ ; \quad \partial_\mu\delta\phi = \partial_\mu\chi_1(x)\ ; \quad \mu = 0,1,2\,, \\ \partial_r\delta\psi &= \frac{1}{\bar\Theta^2}[(\bar\Theta\delta\dot\xi_2 - \dot{\bar\Theta}\delta\xi_2) + (\dot{\bar\Theta}\delta\Theta - \bar\Theta\delta\dot\Theta)]\chi_2 t(x)\ ; \quad \partial_\mu\delta\psi = \partial_\mu\chi_2(x)\ ; \quad \mu = 0,1,2\,. \end{aligned} \tag{3.49}$$

Now we integrate equation (3.23) along r from $0,\infty$ and along the internal unit sphere $\tilde\Omega_2$. The interesting term is the part of the kinetic term involving derivatives along r. After integration by parts it boils down to a mass term for the $1+2$ dimensional fields χ_1, χ_2. Explicitly:

$$\begin{aligned} \int dr d\tilde\Omega_2 \frac{1}{2}\frac{g(r)l^2}{1+l'^2}\partial_r\delta\phi\partial_r\delta\phi &= -\int dr d\tilde\Omega_2\frac{1}{2}\partial_r\left(\frac{g(r)l^2}{1+l'^2}\partial_r\delta\phi\right)\delta\phi = \\ &= -4\pi\left[(\bar\Theta\delta\dot\xi_1 - \dot{\bar\Theta}\delta\xi_1) + (\dot{\bar\Theta}\delta\Theta - \bar\Theta\delta\dot\Theta)\right]\Big|_0^\infty \frac{1}{2}\chi_1^2 = 4\pi m c\frac{1}{2}\chi_1^2\ . \end{aligned} \tag{3.50}$$

Here we have used the same arguments as in equation (3.44). It is clear that one can perform an analogous calculation for the term involving $\partial_r\delta\psi$. The rest of the terms are dealt with straightforwardly by integrating along r. The resulting action is:

$$\frac{S_{\mathrm{eff}}}{(2\pi\alpha')^2} = \int d^3x\left\{\frac{f_{\pi||}^2}{4}\partial_0\chi^*\partial_0\chi - \frac{f_{\pi\perp}^2}{4}\partial_i\chi^*\partial_i\chi - \mu\frac{i}{2}(\chi\partial_0\chi^* - \chi^*\partial_0\chi) + \frac{m_q}{2}\langle\bar\Psi\Psi\rangle_0\chi^*\chi\right\}\,, \tag{3.51}$$

where we have defined a complex scalar field $\chi = \chi_1 + i\chi_2$. The constants in the effective action are given by:

$$\begin{aligned} &\frac{f_{\pi||}^2}{4} = \frac{\mathcal{N}}{2}\int\limits_0^\infty dr\frac{g(r)R^4l^2}{(r^2+l^2)^2}\ ; \quad \frac{f_{\pi\perp}^2}{4} = \frac{\mathcal{N}}{2}\int\limits_0^\infty dr\frac{g(r)R^4l^2}{(r^2+l^2)^2 + R^4H^2}\,, \\ &\mu = \frac{\mathcal{N}}{8}\pi R^4 H; \qquad \langle\bar\psi\psi\rangle = -(2\pi\alpha')\mathcal{N}c_{\mathrm{cr}}\ ; \qquad \mathcal{N} = 4\pi N_f\frac{\mu_5}{g_s}; \qquad m_q = \frac{m}{2\pi\alpha'}\ . \end{aligned} \tag{3.52}$$

The effective action (3.51) is very similar to the one considered in ref. [30], where the author studied Goldstone bosons in linear sigma models with chemical potential, only we have further broken the Lorentz symmetry by the introduction of an external magnetic field. The peculiar feature of this effective Lagrangian is the single time derivative term that is responsible for the unusual quadratic dispersion relation. In ref. [30] the authors have shown that the number of goldstone modes with quadratic dispersion relation is half the number of broken generators. This is exactly what we observe here (two broken generators but only a single goldstone mode).

On the other hand, for the pseudo-Goldstone mode $\omega \to 0$ and to leading order the effective action (3.51) can be written as:

$$S^f_{\text{eff}} = \int d^3x \left\{ \frac{1}{2}\Sigma\epsilon_{ab}\partial_0 U^a U^b - \Sigma\beta h \frac{1}{2} U^{a2} - \frac{1}{2}F^2 \partial_i U^a \partial_i U^a \right\} ; \qquad a = 1 \,\text{or}\, 2\,, \tag{3.53}$$

where

$$U^a = (2\pi\alpha')\chi^a ; \qquad \Sigma = 2\mu; \qquad F^2 = \frac{f^2_{\pi\perp}}{2} ; \qquad \beta h = -\frac{m_q\langle\bar{\psi}\psi\rangle}{2\mu} = \tilde{c}_{\text{cr}}\tilde{m}\frac{4\sqrt{H}}{\pi R} . \tag{3.54}$$

Equation (3.53) corresponds to the effective action describing a ferromagnet in an external magnetic field h with a spontaneous magnetization Σ and a magnetic coupling β [11]. Here $\vec{U} = (U^1, U^2, U^3)$ is a unit vector corresponding to the direction of the spontaneous magnetization of the ferromagnet, and the action (3.53) is describing quadratic fluctuations of the magnetization near the ordered state $U^3 = 1$. The fact that the effective external magnetic field h is proportional to the bare mass $\tilde{m}$ is to be expected since in both descriptions these are the small parameters that reduce the exact global symmetry to an approximate one by coupling to the corresponding order parameters (magnetization and quark condensate corresponding). If one takes into account the re-definitions from equation (3.54) and the definitions of $\tilde{\omega}$ and $\tilde{\vec{k}}$, it is straightforward to check that the dispersion relation of ferromagnetic spin waves [11]:

$$\omega = \gamma k^2 + \beta h ; \qquad \gamma = \frac{F^2}{\Sigma}\,, \tag{3.55}$$

is exactly that of equation (3.35).

Of course many different microscopic systems exhibit the same low energy behavior and hence are described by the same effective Lagrangian. Furthermore the fact that the mass generation process in the D3/D5 system is associated to precisely the same global symmetry breaking pattern (SO(3) $\to$ SO(2)) as the transition from paramagnetic to ferromagnetic phase is also very suggestive. However it is the peculiar single derivative term coming from the Wess-Zumino contribution that is responsible for the observed dispersion relation. Here we have not investigated how far one can go in describing properties of ferromagnets using the D3/D5 set up. It would be interesting to study interaction terms, which would require expanding the effective action beyond quadratic order. In any case it is somewhat satisfying that the dispersion relation of pseudo-Goldstone modes can be related to a real condensed matter phenomenon such as magnon spin waves.

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4 Universal properties of magnetic catalysis in Dp/Dq systems

In this section we focus on some universal features of the mechanism of spontaneous symmetry breaking in an external magnetic field in the context of the Dp/Dq system. In particular we explore the observed discrete self-similar behavior of the theory in the vicinity of the trivial embedding $l \equiv 0$ corresponding to the non-symmetry breaking phase. As we mentioned in section 2 and section 3, for the D3/D7 and the D3/D5 systems, this embedding is unstable and the instability is manifested as a multi-valuedness of the equation of state in the condensate versus bare mass plane $(\tilde{m}, \tilde{c})$ seeded by a logarithmic spiral structure (see figure 2 and figure 5). This spiral structure has been explored in details in ref. [5] in the case of the D3/D7 set up. It has been shown that the spectrum of meson-like excitations also exhibits discrete self-similar structure in the tachyonic sector of the theory. It is interesting that the same structure appears for other first order phase transition in the Dp/Dq set up, such as the meson melting [32] and electrically driven insulator/conductor phase transitions [33, 34]. In ref. [32] it was pointed out that the critical exponents (or more appropriately "scaling exponents") characterizing the logarithmic structure exhibit universal properties and depend only on the dimension of the internal S^n sphere wrapped by the Dq-brane. A similar analysis was performed in ref. [34] for the case of electrically and R-charge chemical potential driven phase transitions and another set of "scaling exponents" was obtained. Here we will extend the analysis of the spiral structure in the case of the D3/D7 system performed in ref. [5] to the general case of Dp/Dq systems T-dual to the D3/D7 intersection and will show that the corresponding scaling exponents guarantee the existence of a discrete self-similar behavior in all Dp/Dq systems of potential phenomenological interest. This suggests the universal role of the external magnetic field as a strong catalyst of mass generation.

To begin with, let us consider the zero temperature Dp-brane solution, given by:

$$\begin{aligned} ds^2 &= K_p^{-\frac{1}{2}} \left(-dt^2 + \sum_{i=1}^{p} dx_i^2 \right) + K_p^{\frac{1}{2}} \left(du^2 + u^2 d\Omega_{8-p}^2 \right) , \\ e^{\Phi} &= g_s K_p^{(3-p)/4} \, ; \qquad C_{01\dots p} = K_p^{-1} \, , \end{aligned} \tag{4.1}$$

where $K_p(u) = (R/u)^{7-p}$ and R is a length scale (the AdS radius in the $p = 3$ case). Now if we introduce a Dq-brane probe having d common space-like directions with the Dp-brane, wrapping an internal $S^n \subset S^{8-p}$ and extended along the holographic coordinate u, we will introduce fundamental matter to the dual gauge theory that propagates along a $(d+1)$-dimensional defect.

Next we parameterize the transverse $9-p$ plane $du^2 + u^2 d\Omega_{8-p}^2$ by:

$$d\rho^2 + dL^2 + \rho^2 d\Omega_n^2 + L^2 d\Omega_{7-p-n}^2 \, , \tag{4.2}$$

where $d\Omega_m^2$ is the metric on a unit radius m-sphere and $\rho^2 + L^2 = u^2$. We also introduce an external magnetic field $H/2\pi\alpha'$, corresponding to the $F_{p-1,p}$ component of the field strength tensor, by fixing a constant B-field in the (x_{p-1}, x_p) plane:

$$B_{(2)} = H dx_{p-1} \wedge dx_p \, . \tag{4.3}$$

Then the DBI part of the Lagrangian governing the classical embedding of the probe is given by:[1]

$$\mathcal{L} \propto e^{-\Phi}\sqrt{-|g_{\alpha\beta}|} = \frac{\sqrt{|\Omega_n|}}{g_s}\rho^n\sqrt{1+L'^2}\sqrt{1+\frac{H^2R^{7-p}}{(\rho^2+L^2)^{\frac{7-p}{2}}}}\ . \tag{4.4}$$

The equation of motion for the classical Dq-brane embedding is given by:

$$\partial_\rho\left(\frac{\rho^n L'}{\sqrt{1+L'^2}}\sqrt{1+\frac{H^2R^{7-p}}{(\rho^2+L^2)^{\frac{7-p}{2}}}}\right)+\frac{7-p}{2}\frac{\rho^n\sqrt{1+L'^2}}{(\rho^2+L^2)^{\frac{9-p}{2}}}\frac{LH^2R^{7-p}}{\sqrt{1+\frac{H^2R^{7-p}}{(\rho^2+L^2)^{\frac{7-p}{2}}}}}=0\ . \tag{4.5}$$

For large $\rho\to\infty$ the second term in equation (4.5) vanishes and the solution $L(\rho)$ has the asymptotic behavior:

$$L(\rho) = m + \frac{c}{\rho^{n-1}} + \cdots\ , \tag{4.6}$$

which encodes [14, 16] the bare quark mass $m_q = m/2\pi\alpha'$ and the quark bilinear condensate $\langle\bar\psi\psi\rangle \propto -c$ of the dual gauge theory.

It is also clear that the equation of motion (4.5) has a trivial solution $L(\rho)\equiv 0$, which preserves the rotational symmetry in the $8-p-n$ plane transverse to both the Dp and Dq-branes. This solution has zero bare quark mass and corresponds to the non-symmetry breaking phase of the dual gauge theory. The solutions in the vicinity of $L\equiv 0$ are unstable and correspond to the interior of the spiral structure that we are studying. In order to obtain the scaling exponents characterizing the spiral we will zoom in on the region close to the origin of the (ρ, L) plane. We first introduce dimensionless variables *via*:

$$\rho = \tilde\rho R H^{\frac{2}{7-p}}\ ; \qquad L = \tilde L R H^{\frac{2}{7-p}}\ ; \qquad \tilde m = m R H^{\frac{2}{7-p}}; \qquad c = \tilde c R^n H^{\frac{2n}{7-p}}\ ; \tag{4.7}$$

and now rescale:

$$\tilde\rho = \lambda\hat\rho\ ; \qquad \tilde L = \lambda\hat L\ . \tag{4.8}$$

In the limit $\lambda\to 0$ equation (4.5) becomes:

$$\partial_{\hat\rho}\left(\frac{\hat\rho^n}{(\hat\rho^2+\hat L^2)^{\frac{7-p}{4}}}\frac{\hat L'}{\sqrt{1+\hat L'^2}}\right)+\frac{7-p}{2}\sqrt{1+\hat L'^2}\frac{\hat\rho^n\hat L}{(\hat\rho^2+\hat L^2)^{\frac{11-p}{4}}}=0\ . \tag{4.9}$$

The solutions to equation (4.9) have the scaling property that if $\hat L(\hat\rho)$ is a solution, then so is $\frac{1}{\mu}\hat L(\mu\hat\rho)$. In order to explore the vicinity of the critical $\hat L\equiv 0$ solution we define $\hat L = 0+\zeta(\hat\rho)$ and linearize with respect to ζ, the result is:

$$\partial_{\hat\rho}\left(\hat\rho^{n-\frac{7-p}{2}}\zeta'\right)+\frac{7-p}{2}\hat\rho^{n-\frac{11-p}{2}}\zeta=0\ . \tag{4.10}$$

Next we look for solutions of equation (4.10) of the form $\zeta = \hat\rho^\nu$. The quadratic equation for ν that we obtain is:

$$2\nu^2+(n+d-6)\nu+(n-d+4)=0\ . \tag{4.11}$$

[1]We consider only systems T-dual to the D3/D7 one, which imposes the constraint $p-d+n+1=4$.

We have used the constraint $p = 3+d-n$. Now in order to have a logarithmic spiral (which seeds the multi-valuedness of the equation state) we need to have two complex roots. The condition for that is:

$$(n + d - 6)^2 < 8(n - d + 4) \ . \tag{4.12}$$

Note that in order to be able to turn on a magnetic field we need $d \geq 2$. In addition we are not interested in theories with $d > 3$. It is then easy to check that for all possible values of n (clearly $n < 5$) the condition (4.12) is satisfied. Then the roots of equation (4.11) $\nu_\pm$ are given by:

$$\nu_\pm = -r_{n,d} \pm i\alpha_{n,d} \ ; \quad r_{n,d} = \frac{n+d-6}{4} \geq -\frac{3}{4} \ ; \quad \alpha_{n,d} = \frac{1}{4}\sqrt{8(n-d+4)-(n+d-6)^2} \ . \tag{4.13}$$

The inequality in the second formula in equation (4.13) is saturated for the minimum possible values $(n, d) = (1, 2)$. The most general solution of equation (4.10) can then be written as:

$$\zeta(\hat\rho) = \frac{1}{\hat\rho^{r_{n,d}}}(A\cos(\alpha_{n,d}\ln\hat\rho) + B\sin(\alpha_{n,d}\ln\hat\rho)) \ . \tag{4.14}$$

Now the scaling property of equation (4.9) suggests the following transformation of the parameters (A, B) under re-scaling of the initial condition $\hat L(0) \equiv L_0 \to \frac{1}{\mu}\hat L_0$:

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \frac{1}{\mu^{r_n+1}} \begin{pmatrix} \cos(\alpha_n \ln\mu) & \sin(\alpha_n\ln\mu) \\ -\sin(\alpha_n\ln\mu) & \cos(\alpha_n\ln\mu) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \ . \tag{4.15}$$

For a fixed choice of the parameters A and B, the parameters (A', B') describe a logarithmic spiral, whose step and periodicity are set by the real and imaginary parts of the critical/scaling exponents $r_{n,d}$ and $\alpha_{n,d}$. Note that from the inequality in equation (4.13) it follows that $r_{n,d} + 1 \geq \frac{1}{4} > 0$ and hence the spiral is revolving as one scales away from the critical $\hat L \equiv 0$ solution.

This self-similar structure of the embeddings near the critical solution $\hat L \equiv 0$ in our zoomed in region parameterized by $(\hat\rho, \hat L)$ is transferred by a linear transformation to the structure of the solutions in the (m, c) parameter space. The parameters corresponding to the critical $L \equiv 0$ embedding are given by $(0, 0)$. Then sufficiently close to the critical embedding we can expand:

$$\begin{pmatrix} m \\ c \end{pmatrix} = M \begin{pmatrix} A \\ B \end{pmatrix} + \mathcal{O}(A^2, B^2, AB) \ . \tag{4.16}$$

The constant matrix M depends on the properties of the system. Generically it should be invertible (numerically we have verified that this is the case) and therefore in the vicinity of the parameter space close to the critical embedding (m, c) there is a discrete self-similar structure determined by the transformation:

$$\begin{pmatrix} m' \\ c' \end{pmatrix} = \frac{1}{\mu^{r_n+1}} M \begin{pmatrix} \cos(\alpha_n \ln\mu) & \sin(\alpha_n\ln\mu) \\ -\sin(\alpha_n\ln\mu) & \cos(\alpha_n\ln\mu) \end{pmatrix} M^{-1} \begin{pmatrix} m \\ c \end{pmatrix} \ . \tag{4.17}$$

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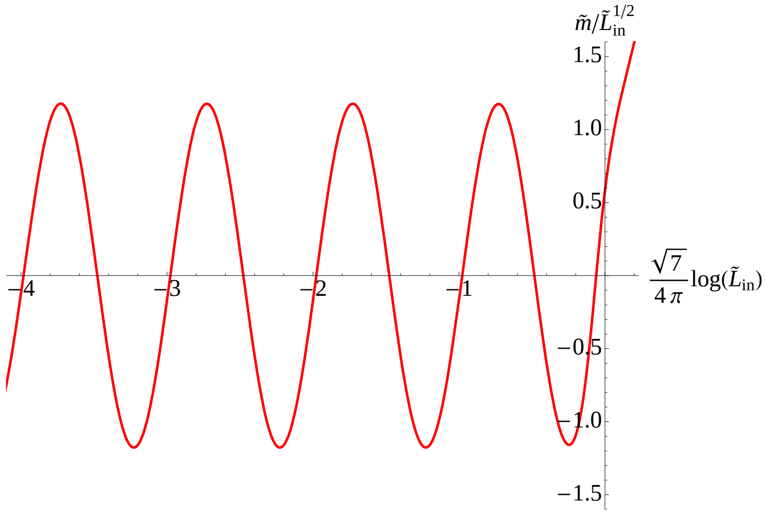

Figure 11. A plot of $\frac{\sqrt{7}}{4\pi}\log\tilde{L}_{\rm in}$ *vs.* $\tilde{m}/\tilde{L}_{\rm in}^{1/2}$. For sufficiently small $\tilde{L}_{\rm in}$ the curve is an harmonic function of unit period.

Note that the linear map corresponding to the constant matrix M would rotate, stretch and/or shrink (along the different axes) the spiral defined *via* the transformation (4.15). However the overall shape of the curve defined *via* equation (4.17) still remains a spiral revolving around the origin of the $(\tilde{m}, \tilde{c})$ plane (see figure 5 for the case of the D3/D7 system). This suggests that the state corresponding to the center of the spiral (the $L \equiv 0$ solution) is unstable and hence there is a dynamical mass generation in the theory. (The stable state at zero bare quark mass has a non-zero condensate) Therefore we learn that for all Dp/Dq systems T-dual to the D3/D7 intersection (and with $d \geq 2$ so that a magnetic field can be switched on) the effect of the magnetic field is to break a global internal symmetry and generate a dynamical mass.

To conclude this discussion we will provide a numerical check of the consistency of our analysis. To this end we consider the separation of the Dq and Dp branes at $L_{\rm in} \equiv L(0)$ (note that $L_{\rm in}$ is proportional to the dynamically generated quark mass). Now if we start from some $L_{\rm in}^0$ and transform to $L_{\rm in} = \frac{1}{\mu}L_{\rm in}^0$, we can solve for μ and generate a parametric plot of $\tilde{m}/(\tilde{L}_{\rm in})^{r_{n,d}+1}$ *vs.* $\alpha_{n,d}\log\tilde{L}_{\rm in}/2\pi$. The transformation (4.17) requires that the resulting plot be an harmonic function of unit period. For the particular case of the D3/D5 system we have $r_{2,2} = -1/2$ and $\alpha_{2,2} = \sqrt{7}/2$. The corresponding plot is presented in figure 11. For sufficiently small $\tilde{L}_{\rm in}$ the plot is indeed an harmonic function of unit period.

5 Conclusion

In this paper we have investigated the properties of strongly coupled, large N_c gauge theories in the presence of an external magnetic field. Both in three and four dimensions, the holographic approach reproduces the behavior expected from classical field theory arguments and the magnetic catalysis of global symmetry breaking is shown to be a universal

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feature of a family of strongly coupled gauge theories. This is another success for the internal consistency of the gauge/gravity duality.

As has now become commonplace in such studies, the spectrum of mesonic states is directly identifiable by numerical methods. Moreover, in the present study we are able to make a large number of statements analytically by studying the systems in the chiral limit. Indeed in this regime, in which the global symmetry is broken dynamically due to the presence of the magnetic field the results are almost completely tractable analytically.

In $3+1$ dimensions we are able to show that the Gell-Mann–Oakes–Renner relation holds exactly and in this setting where a subgroup of the special relativistic transformations remains unbroken in the presence of the magnetic field, a relativistic dispersion relation is indeed recovered. In the $2+1$ dimensional case however, all trace of the boost invariance is lost once the magnetic field is turned on. The counting of Goldstone modes then becomes more subtle but we are able to show that this holographic setup gives the correct number of massless modes expected from the non-relativistic Goldstone counting rules. In addition we are able to show that these modes obey a quadratic dispersion relation, in contrast to the relativistic case in one spatial dimension higher.

In the present investigation we have focused on the low energy chiral lagrangian, calculated up to quadratic order in the goldstone mode excitations. In the $2+1$ dimensional setting we were able to show explicitly that at this order our system reproduces the low energy behavior of spin-wave excitations in a ferromagnet. The matching of symmetry breaking patterns between the two systems is the root of this equality. It seems unlikely however that such an agreement will hold to higher order. It would certainly be possible to investigate the systems discussed here at higher order in the low energy degrees of freedom and this would be an interesting direction for future work.

The study of flavor degrees of freedom in a wide range of conditions is certainly well worth pursuing further using the methods illustrated in this paper. The magnitude of the phase-space involved with studying flavor, both abelian and non-abelian, in the presence of external electric fields and magnetic fields, temperature and finite chemical potential in a variety of dimensions means that there is surely plenty more to be found in even the simplest systems.

In the case of $2+1$ dimensions, the understanding of gauge theories in the presence of low temperature and high magnetic field is of much interest. In particular this is one area which may be accessible experimentally and one may be able to gain insight into such intriguing phenomena as the quantum hall effect. It seems very likely that we will make further inroads into understanding such effects using the AdS/CFT correspondence in the not too distant future.

The diversity of phenomena that we can investigate using holographic techniques is clearly far larger than was expected in the early days of the AdS/CFT conjecture. The prospects of obtaining deep insight into such fascinating systems as non-conventional superconductors, the quantum hall effect and strongly coupled plasmas are real and exciting and the community continues to make progress in these directions.

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A Low energy effective action for Φ

Let us begin with the Lagrangian for the quadratic fluctuations equation (2.11)

$$\begin{aligned}
\mathcal{L}_{\Phi\Phi} &= -(2\pi\alpha')^2\frac{\mu_7}{g_s}\frac{1}{2}\sqrt{|g_{S^3}|}\frac{gR^2L_0^2}{\rho^2+L_0^2}S^{ab}\partial_a\Phi\partial_b\Phi\,,\\
\mathcal{L}_{\Phi A} &= -(2\pi\alpha')^2\frac{\mu_7}{g_s}\sqrt{|g_{S^3}|}H\partial_\rho K\Phi F_{01}\,,\\
\mathcal{L}_{AA} &= -(2\pi\alpha')^2\frac{\mu_7}{g_s}\sqrt{|g_{S^3}|}\frac{1}{4}gS^{aa'}S^{bb'}F_{ab}F_{a'b'}\ .
\end{aligned} \tag{A.1}$$

In order to obtain an effective action for the fluctuations along Φ we will integrate out the fluctuations of the gauge field and in particular the A_0, A_1 components. In more detail the contribution from the last two equations in equation (A.1) can be written as:

$$\begin{aligned}
\mathcal{L}_{AA}+\mathcal{L}_{\Phi A} \propto\ & \frac{1}{2}gS^{11}S^{aa'}(-\partial_aA_0\partial_{a'}A_0+\partial_aA_1\partial_{a'}A_1)-\frac{1}{2}g(S^{11})^2(\partial_0A_0-\partial_1A_1)^2\\
&+H\partial_\rho K\Phi F_{01}\ .
\end{aligned} \tag{A.2}$$

To integrate out the A_0, A_1 components of the gauge field we simply obtain the equations of motion for A_0 and A_1 and substitute them into the action. The equations of motion are:

$$\begin{aligned}
\partial_a(gS^{11}S^{aa'}\partial_aA_0)+g(S^{11})^2\partial_0(\partial_0A_0-\partial_1A_1)+H\partial_\rho K\partial_1\Phi &= 0\,,\\
\partial_a(gS^{11}S^{aa'}\partial_aA_1)+g(S^{11})^2\partial_1(\partial_0A_0-\partial_1A_1)+H\partial_\rho K\partial_0\Phi &= 0\,,
\end{aligned} \tag{A.3}$$

which can be written as:

$$\begin{aligned}
\partial_a\left(gS^{11}S^{aa'}\partial_aF_{01}\right)-H\partial_\rho K\square_{(1,1)}\Phi = 0\,; \quad \square_{(1,1)}\equiv -\partial_0^2+\partial_1^2\,,\\
\partial_a\left(gS^{11}S^{aa'}\partial_a(\partial_0A_0-\partial_1A_1)\right)-g(S^{11})^2\square_{(1,1)}(\partial_0A_0-\partial_1A_1) = 0\ .
\end{aligned} \tag{A.4}$$

Substituting back into the action (A.2) and integrating by parts leads to:

$$\mathcal{L}_{AA}+\mathcal{L}_{\Phi A}\propto -\frac{1}{2}HK\partial_\rho(\Phi F_{01})\,. \tag{A.5}$$

The equation of motion for F_{01} can be written as:

$$\partial_\rho F_{01} = \frac{HK}{\Psi_1^2}\Box_{(1,1)}\Phi - \frac{1}{\Psi_1^2}\int d\rho g(S^{11})^2\tilde{\Box}F_{01}\,, \tag{A.6}$$

where Ψ_1 is defined in equation (2.22). Now we substitute into the action (A.5) and denote by $\tilde{\Phi}(x)$ the dimensionally reduced field Φ, to obtain:

$$\mathcal{L}_{AA} + \mathcal{L}_{\Phi A} = (2\pi\alpha')^2\frac{\mu_7}{g_s}\sqrt{|g_{S^3}|}\frac{1}{2}\frac{H^2K^2}{\Psi_1^2}\tilde{\Phi}\Box_{(1,1)}\tilde{\Phi} + \dots\,, \tag{A.7}$$

where we have ignored higher order derivatives terms. Combining this with the dimensionally reduced term $\mathcal{L}_{\Phi\Phi}$ we obtain the result:

$$\mathcal{L} \propto \frac{1}{2}\left(\nu\Psi^2 + \frac{H^2K^2}{\Psi_1^2}\right)\left[-(\partial_0\tilde{\Phi})^2 + (\partial_1\tilde{\Phi})^2\right] + \frac{1}{2}\tilde{\nu}\Psi^2\left[(\partial_2\tilde{\Phi})^2 + (\partial_3\tilde{\Phi})^2\right] + \dots. \tag{A.8}$$

Where $\nu, \tilde{\nu}$ and Ψ are defined in equation (2.22). In order to obtain a mass term for the dimensionally reduced field $\tilde{\Phi}$ we have to take into account the radial dependence of the field Φ. Our analysis from section 2.2.1 suggests that we should consider the following ansatz:

$$\Phi(\rho, x) = \frac{\psi(\rho)}{\Psi(\rho)}\tilde{\Phi}(x)\,, \tag{A.9}$$

where Ψ is defined in equation (2.22) and we require that for the spontaneous symmetry breaking classical embedding (denoted by $\bar{L}_0$) we have that $\psi|_{\bar{L}_0} = \Psi|_{\bar{L}_0} \equiv \bar{\Psi}$. Then if we consider embeddings in the vicinity of $\bar{L}_0$ corresponding to small bare quark mass δm we can expand:

$$\psi = \bar{\Psi} + \delta\psi\ ; \qquad \Phi(\rho, x) = \left[1 + \delta\left(\frac{\psi}{\Psi}\right)\right]\tilde{\Phi}(x)\ . \tag{A.10}$$

Now if we demand that as $\delta m \to 0$ we have small momenta and a small mass term (which vanish at the critical embedding) to leading order we still have the expression from equation (A.8) plus some small mass term involving derivatives along ρ:

$$\begin{aligned}\mathcal{L} \propto\ & \frac{1}{2}\left(\bar{\nu}\bar{\Psi}^2 + \frac{H^2\bar{K}^2}{\bar{\Psi}_1^2}\right)\left[-(\partial_0\tilde{\Phi})^2 + (\partial_1\tilde{\Phi})^2\right] + \frac{1}{2}\bar{\tilde{\nu}}\bar{\Psi}^2\left[(\partial_2\tilde{\Phi})^2 + (\partial_3\tilde{\Phi})^2\right]\\ & -\frac{1}{2}\partial_\rho\left[\bar{\Psi}^2\partial_\rho\delta\left(\frac{\psi}{\Psi}\right)\right]\tilde{\Phi}^2 + \dots\,,\end{aligned} \tag{A.11}$$

where we have integrated by parts the last term and the dots represent higher derivatives terms and other sub-leading terms. Now it is straightforward to integrate along the unit S^3 and the radial coordinate ρ. Let us provide some more details in the integration of the mass term:

$$\int\limits_0^\infty d\rho\partial_\rho\left[\bar{\Psi}^2\partial_\rho\delta\left(\frac{\psi}{\Psi}\right)\right]\tilde{\Phi}^2 = \left[(\bar{\Psi}\delta\psi' - \delta\psi\bar{\Psi}') + (\delta\Psi\bar{\Psi}' - \delta\Psi'\bar{\Psi})\right]\tilde{\Phi}^2\Big|_0^\infty = -2c\delta m\tilde{\Phi}^2\,. \tag{A.12}$$

Then for the final form of the effective action one obtains:

$$S_{\text{eff}} = -\mathcal{N}\int d^4x\left\{\left[-(\partial_0\tilde{\Phi})^2 + (\partial_1\tilde{\Phi})^2\right] + \gamma\left[(\partial_2\tilde{\Phi})^2 + (\partial_3\tilde{\Phi})^2\right] - \frac{2\langle\bar{\psi}\psi\rangle}{f_\pi^2}m_q\tilde{\Phi}^2\right\} + \dots\,, \tag{A.13}$$

where:

$$\mathcal{N} = (2\pi\alpha')^2 N_f \frac{\mu_7}{g_s}\pi^2 \int_0^\infty d\rho \left[\bar{\nu}\bar{\Psi}^2 + \frac{H^2\bar{K}^2}{\bar{\Psi}_1^2}\right]; \qquad f_\pi^2 = \frac{4\mathcal{N}}{(2\pi\alpha')^2}; \qquad m_q = \frac{\delta m}{2\pi\alpha'}, \quad \text{(A.14)}$$

$$\gamma = \int_0^\infty d\rho \left(\bar{\nu}\bar{\Psi}^2\right) \Big/ \int_0^\infty d\rho \left(\bar{\nu}\bar{\Psi}^2 + \frac{H^2\bar{K}^2}{\bar{\Psi}_1{}^2}\right); \quad \langle\bar{\psi}\psi\rangle = -\frac{N_f}{(2\pi\alpha')^3}\frac{c}{2\pi g_s}. \quad \text{(A.15)}$$

One can see that this is the most general quadratic action consistent with the $\mathrm{SO}(1,1)\times\mathrm{SO}(2)$ space-time symmetry. Furthermore the explicit form of the mass term is in accord with the Gell-Mann–Oakes–Renner relation (2.37). To obtain the expression for f_π^2 provided in equation (A.14) one needs to consider the strict $m_q \to 0$ limit and use that in this limit $\tilde{\Phi} = \phi/(2\pi\alpha')$. Next since ϕ corresponds to rotations in the transverse $\mathbb{R}^2$ plane and is thus the angle of chiral rotation [16], the normalization of the kinetic term in the effective action (A.13) is given by $\mathcal{N} = (2\pi\alpha')^2 f_\pi^2/4$. The last relation determines f_π^2 in terms of $\mathcal{N}$.

References

[1] J.D. Edelstein, J.P. Shock and D. Zoakos, *The AdS/CFT correspondence and non-perturbative QCD*, *AIP Conf. Proc.* **1116** (2009) 265 [arXiv:0901.2534] [SPIRES].

[2] S.S. Gubser and A. Karch, *From gauge-string duality to strong interactions: a pedestrian's guide*, arXiv:0901.0935 [SPIRES].

[3] E. Keski-Vakkuri and P. Kraus, *Effective theory of the quantum Hall effect in AdS/CFT*, *JHEP* **09** (2008) 130 [arXiv:0805.4643] [SPIRES].

[4] S.A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, arXiv:0903.3246 [SPIRES].

[5] V.G. Filev, C.V. Johnson, R.C. Rashkov and K.S. Viswanathan, *Flavoured large-N gauge theory in an external magnetic field*, *JHEP* **10** (2007) 019 [hep-th/0701001] [SPIRES].

[6] V.P. Gusynin, V.A. Miransky and I.A. Shovkovy, *Catalysis of dynamical flavor symmetry breaking by a magnetic field in (2+1)-dimensions*, *Phys. Rev. Lett.* **73** (1994) 3499 [*Erratum ibid.* **76** (1996) 1005] [hep-ph/9405262] [SPIRES].

[7] V.P. Gusynin, V.A. Miransky and I.A. Shovkovy, *Dimensional reduction and dynamical chiral symmetry breaking by a magnetic field in (3+1)-dimensions*, *Phys. Lett.* **B 349** (1995) 477 [hep-ph/9412257] [SPIRES].

[8] K.G. Klimenko, *Three-dimensional Gross-Neveu model in an external magnetic field*, *Theor. Math. Phys.* **89** (1992) 1161 [*Teor. Mat. Fiz.* **89** (1991) 211] [SPIRES].

[9] K.G. Klimenko, *Three-dimensional Gross-Neveu model at nonzero temperature and in an external magnetic field*, *Z. Phys.* **C 54** (1992) 323 [SPIRES].

[10] K.G. Klimenko, *Three-dimensional Gross-Neveu model at nonzero temperature and in an external magnetic field*, *Theor. Math. Phys.* **90** (1992) 1 [*Teor. Mat. Fiz.* **90** (1992) 3] [SPIRES].

[11] C.P. Hofmann, *Spin-wave scattering in the effective Lagrangian perspective*, *Phys. Rev.* **B 60** (1999) 388 [cond-mat/9805277] [SPIRES].

[12] V.G. Filev, *Criticality, scaling and chiral symmetry breaking in external magnetic field*, *JHEP* **04** (2008) 088 [arXiv:0706.3811] [SPIRES].

[13] J. Erdmenger, R. Meyer and J.P. Shock, *AdS/CFT with flavour in electric and magnetic Kalb-Ramond fields*, *JHEP* **12** (2007) 091 [arXiv:0709.1551] [SPIRES].

[14] A. Karch and E. Katz, *Adding flavor to AdS/CFT*, *JHEP* **06** (2002) 043 [hep-th/0205236] [SPIRES].

[15] M. Kruczenski, D. Mateos, R.C. Myers and D.J. Winters, *Meson spectroscopy in AdS/CFT with flavour*, *JHEP* **07** (2003) 049 [hep-th/0304032] [SPIRES].

[16] M. Kruczenski, D. Mateos, R.C. Myers and D.J. Winters, *Towards a holographic dual of large-N_c QCD*, *JHEP* **05** (2004) 041 [hep-th/0311270] [SPIRES];

[17] J. Babington, J. Erdmenger, N.J. Evans, Z. Guralnik and I. Kirsch, *Chiral symmetry breaking and pions in non-supersymmetric gauge/gravity duals*, *Phys. Rev.* **D 69** (2004) 066007 [hep-th/0306018] [SPIRES].

[18] N.J. Evans and J.P. Shock, *Chiral dynamics from AdS space*, *Phys. Rev.* **D 70** (2004) 046002 [hep-th/0403279] [SPIRES].

[19] D. Arean, A. Paredes and A.V. Ramallo, *Adding flavor to the gravity dual of non-commutative gauge theories*, *JHEP* **08** (2005) 017 [hep-th/0505181] [SPIRES].

[20] D. Arean and A.V. Ramallo, *Open string modes at brane intersections*, *JHEP* **04** (2006) 037 [hep-th/0602174] [SPIRES].

[21] R.C. Myers and R.M. Thomson, *Holographic mesons in various dimensions*, *JHEP* **09** (2006) 066 [hep-th/0605017] [SPIRES].

[32] D. Mateos, R.C. Myers and R.M. Thomson, *Holographic phase transitions with fundamental matter*, *Phys. Rev. Lett.* **97** (2006) 091601 [hep-th/0605046] [SPIRES].

[23] M. Gell-Mann, R.J. Oakes and B. Renner, *Behavior of current divergences under* $\mathrm{SU}(3) \times \mathrm{SU}(3)$, *Phys. Rev.* **175** (1968) 2195 [SPIRES].

[24] H.B. Nielsen and S. Chadha, *On how to count Goldstone bosons*, *Nucl. Phys.* **B 105** (1976) 445 [SPIRES].

[25] D. Mateos, R.C. Myers and R.M. Thomson, *Thermodynamics of the brane*, *JHEP* **05** (2007) 067 [hep-th/0701132] [SPIRES].

[26] R.C. Myers, A.O. Starinets and R.M. Thomson, *Holographic spectral functions and diffusion constants for fundamental matter*, *JHEP* **11** (2007) 091 [arXiv:0706.0162] [SPIRES].

[27] N. Evans and E. Threlfall, *R-charge chemical potential in the M2-M5 system*, arXiv:0807.3679 [SPIRES].

[28] A. O'Bannon, *Toward a holographic model of superconducting fermions*, *JHEP* **01** (2009) 074 [arXiv:0811.0198] [SPIRES].

[29] R.C. Myers and M.C. Wapler, *Transport properties of holographic defects*, *JHEP* **12** (2008) 115 [arXiv:0811.0480] [SPIRES].

[30] T. Brauner, *Goldstone boson counting in linear σ-models with chemical potential*, *Phys. Rev.* **D 72** (2005) 076002 [hep-ph/0508011] [SPIRES].

[31] T. Brauner, *Goldstone bosons in presence of charge density*, *Phys. Rev.* **D 75** (2007) 105014 [hep-ph/0701110] [SPIRES].

[32] D. Mateos, R.C. Myers and R.M. Thomson, *Holographic phase transitions with fundamental matter*, *Phys. Rev. Lett.* **97** (2006) 091601 [hep-th/0605046] [SPIRES].

[33] T. Albash, V.G. Filev, C.V. Johnson and A. Kundu, *Quarks in an external electric field in finite temperature large-N gauge theory*, *JHEP* **08** (2008) 092 [arXiv:0709.1554] [SPIRES].

[34] V.G. Filev and C.V. Johnson, *Universality in the large-N_c dynamics of flavour: thermal vs. quantum induced phase transitions*, *JHEP* **10** (2008) 058 [arXiv:0805.1950] [SPIRES].

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